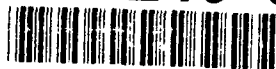


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# Exact Performance of Filtered and Weighted Energy Detector with Mismatched Frequency and Time Locations and Characteristics

Albert H. Nuttall  
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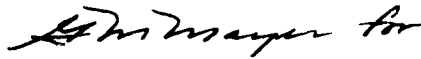
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## PREFACE

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## LIST OF SYMBOLS

$\Delta$	time sampling increment, figure 1
$k, m$	integers denoting discrete time, figure 1 and (1)
$x(k\Delta)$	digital input sequence, figure 1 and (1)
$h(m\Delta)$	digital filter, figure 1 and (5)
$y(k\Delta)$	filter output sequence, figure 1 and (5)
$w(k\Delta)$	summer weights, figure 1, (6) and (7)
$z$	system output, figure 1 and (6)
$n(k\Delta)$	input noise sequence, (1)
$s(k\Delta)$	input signal sequence, (1)
$g(k\Delta)$	signal gating sequence, (1) and (4)
overbar	ensemble average, (2)
$R_n(m\Delta)$	input noise covariance, (2)
$R_s(m\Delta)$	input signal covariance, (3)
$k_a$	input signal start time integer, (4)
$k_b$	input signal stop time integer, (4)
$y_s(k\Delta)$	filter output signal, (5) and (13)
$y_n(k\Delta)$	filter output noise, (5) and (8)
$k_c$	observation start time integer, (7)
$k_d$	observation stop time integer, (7)
$k_1\Delta, k_2\Delta$	general covariance times, (9)
$C_n$	filter output noise covariance, (9)
$\phi_h(j\Delta)$	filter correlation function, (10)
$\sigma_n^2$	input noise power, (11) and (45)
$C_s$	filter output signal covariance, (15)
$K_1, K_2$	auxiliary integers, (18)



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$K_o$	auxiliary integer, (19)
$C_y$	covariance of total filter output, (23)
$a_k$	auxiliary random variables, (24)
$C_a$	covariance of $\{a_k\}$ , (26)
$C$	matrix with elements (26)
$Q$	normalized modal matrix of eigenvectors of $C$ , (27)
$\Lambda$	diagonal matrix of eigenvalues $\{\lambda_k\}$ of $C$ , (27)
$T$	transpose of matrix, (27)
$A$	column vector of $\{a_k\}$ , (28)
$B$	linear transformation of $A$ , (29)
$\{b_k\}$	elements of $B$ , (30)
$f_z(\xi)$	characteristic function of output $z$ , (32)
$\tau$	argument (delay) of correlation functions, (33)
$J$	number of input signal covariance components, (33)
$\alpha_j$	input signal scaling, (33)
$\tau_{sj}$	input signal time constant, (33)
$\sigma_s^2$	input signal power, (34)
$f$	frequency, (35)
$G_s(f)$	input signal spectrum, (35)
$\omega$	$2\pi f$ , (35)
$a_j$	input signal exponential parameter, (36)
$M$	number of input noise covariance components, (44)
$\beta_m$	input noise scaling, (44)
$\tau_{nm}$	input noise time constant, (44)
$b_m$	input noise exponential parameter, (46)



$N$	number of filter components, (47)
$\psi_n$	filter scaling, (47)
$\tau_{hn}$	filter time constant, (47)
$c_n$	filter exponential parameter, (49)
$v_n$	auxiliary filter constant, (51)
$\underline{H}(z)$	filter $z$ transform, (52)
$\oplus$	convolution, (55) and (A-2)
$C(k\Delta)$	auxiliary function, (56)
$\mu_m, \gamma_n$	auxiliary constants, (60)
$S$	auxiliary function, (65) and (69)
$g$	auxiliary function, (68)
$f_0(\xi)$	characteristic function for noise only, (73)
$p_0(u)$	probability density function for noise only, (73)
$\Delta_0$	sampling increment in $\xi$ applied to $f_0(\xi)$ , (73)
$\tilde{p}_0(u)$	approximation to $p_0(u)$ , (73) and (74)
$N_0$	size of FFT for $\tilde{p}_0(u)$ , (74)
$\delta_0$	increment in $u$ for $\tilde{p}_0(u)$ , (74) and (75)
$u_0$	maximum useful $u$ for $\tilde{p}_0(u)$ , (75)
$f_1(\xi)$	characteristic function for signal present, (76)
$p_1(u)$	probability density function for signal present, (76)
$\Delta_1$	sampling increment in $\xi$ applied to $f_1(\xi)$ , (76)
$\tilde{p}_1(u)$	approximation to $p_1(u)$ , (76) and (77)
$N_1$	size of FFT for $\tilde{p}_1(u)$ , (77)
$\delta_1$	increment in $u$ for $\tilde{p}_1(u)$ , (77) and (78)
$u_1$	maximum useful $u$ for $\tilde{p}_1(u)$ , (78)
$E_k(u)$	exceedance distribution corresponding to $f_k(\xi)$ , (80)
$\text{Im}$	imaginary part, (80)

$P_D$	detection probability
$P_F$	false alarm probability
CDF	cumulative distribution function
EDF	exceedance distribution function
R	input signal to noise ratio $\sigma_s^2/\sigma_n^2$ , (82)
$\delta_b(\tau)$	unit impulse train with spacing b, (A-2)

EXACT PERFORMANCE OF FILTERED AND WEIGHTED ENERGY DETECTOR WITH  
MISMATCHED FREQUENCY AND TIME LOCATIONS AND CHARACTERISTICS

INTRODUCTION

The detection of weak random signals of unknown duration, bandwidth, time location, and frequency shift in the presence of colored noise is often accomplished in practice by filtering the received waveform to the frequency band of interest, squaring the filter output, and time-weighting this quantity prior to accumulation and threshold comparison. The performance of this processor obviously depends on the spectra of the received signal and noise processes as well as on the transfer characteristics of the filter employed. Lack of detailed knowledge of the signal spectral characteristics or frequency shift will often dictate that a fairly broad (mismatched) filter passband be utilized in order that the signal be passed when present. Similarly, uncertainty about the signal location or duration will mandate that the receiver observation time be lengthened in order to ensure capture of any impinging signal energy.

Additionally, the received signal process is often characterized as a finite-duration burst of a stationary process. Since the signal time location and duration are often unknown, the receiving filter must be kept open at all times, thereby allowing the filter output noise to reach its full-strength steady state value. By contrast, a gated input signal leads to a filter output component which gradually builds up in time and

then decays slowly after the signal is turned off. These nonstationary effects, coupled with the particular time weighting employed during the observation interval, influence the performance of the detector in a complicated nonobvious fashion.

There is a need to be able to calculate exactly the performance of this type of signal processor in the presence of such deleterious factors, so that the degradations associated with lack of knowledge can be ascertained quantitatively. Furthermore, it will not suffice to resort to Gaussian approximations for the processor output, because the number of samples employed are not large enough to utilize the central limit theorem, especially on the tails of the distribution (small false alarm probabilities) where we are interested. This need is addressed in this report for the case of Gaussian input signals in Gaussian noise, with the result that programs are furnished for exactly assessing the performance of the mismatched energy detector for a very wide variety of characteristics and selections of parameters. In particular, no presumptions are made about the input signal time-bandwidth product or about the size of the product of observation time and filter bandwidth.

## SYSTEM DESCRIPTION AND ASSUMPTIONS

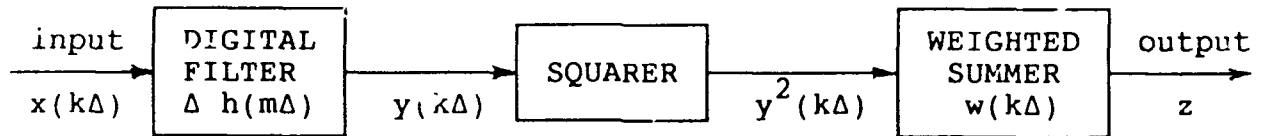


Figure 1. Block Diagram of Detector

The detector of interest is the digital processor depicted in figure 1; time sampling increment  $\Delta$  is arbitrary but should be approximately matched to the coherence times of the input signal and filter. The digital input sequence  $x(k\Delta)$ , filter impulse response  $\Delta h(m\Delta)$ , and summer weights  $w(k\Delta)$  are all real. Input  $x(k\Delta)$  consists of either noise-alone or gated signal plus noise, according to model

$$x(k\Delta) = \left\{ \begin{array}{c} n(k\Delta) \\ \text{or} \\ s(k\Delta) g(k\Delta) + n(k\Delta) \end{array} \right\} \quad \text{for all } k. \quad (1)$$

Input noise  $n(k\Delta)$  is present for all time and is a stationary zero-mean discrete Gaussian sequence with covariance

$$R_n(m\Delta) = \overline{n(k\Delta) n(k\Delta - m\Delta)} \quad \text{for all } m, k, \quad (2)$$

where an overbar denotes an ensemble average. The numerical evaluation of the values of the noise covariance, directly from a specified noise spectrum  $G_n(f)$ , is considered in appendix A.

Underlying input signal  $s(k\Delta)$  in (1) (if present) is also a

stationary zero-mean discrete Gaussian sequence with covariance

$$R_s(m\Delta) = \overline{s(k\Delta) s(k\Delta - m\Delta)} \quad \text{for all } m, k. \quad (3)$$

However, input signal samples  $s(k\Delta)$  are gated by function  $g(k\Delta)$  which is nonzero only in a limited time region:

$$g(k\Delta) = 1 \quad \text{for } k_a \leq k \leq k_b, \text{ zero otherwise.} \quad (4)$$

This results in a gated burst of stationary signal sequence  $s(k\Delta)$  being inputted to digital filter  $h(m\Delta)$ ; the input signal starting and ending times  $k_a\Delta$  and  $k_b\Delta$ , respectively, are generally unknown, except in an approximate way. This generality allows for consideration of input signals of unknown arrival time and duration at the detector input.

The filter output  $y(k\Delta)$  can be conveniently broken into signal and noise components, in accordance with (1), and is available by means of discrete convolution

$$y(k\Delta) = \sum_m \Delta h(m\Delta) x(k\Delta - m\Delta) = y_s(k\Delta) + y_n(k\Delta), \quad (5)$$

when an input signal is present. (Summations without limits are over  $(-\infty, +\infty)$ .) However, due to the gating in (1), filter output signal  $y_s(k\Delta)$  will have transient phases, including a buildup just after time  $k_a\Delta$  and a decay just after time  $k_b\Delta$ . This nonstationary behavior is included and exactly accounted for in the following analysis.

The filter output (5) is then squared, scaled by weights  $w(k\Delta)$ , and summed to give system output

$$z = \sum_k w(k\Delta) y^2(k\Delta) , \quad (6)$$

where the weights are nonzero only in a limited observation time specified by

$$w(k\Delta) \neq 0 \quad \text{for } k_c \leq k \leq k_d . \quad (7)$$

These weights need not be uniform; for example, they could be exponential. Output  $z$  is compared with a threshold for a declaration of signal absent versus signal present.

Observation time limits  $k_c\Delta$  and  $k_d\Delta$  are under the control of the receiver processor, and would hopefully match fairly well the time interval over which the gated input signal is received; see (4). But, in any event, the degree of generality in (7) allows for investigation of observation times that might be too short, thereby losing some signal energy, or too long, thereby picking up additional unwanted noise components; both situations lead to deterioration of the detectability of weak signals and should be avoided if possible. An illustration of the parameters is given in figure 2, for signal-only present.

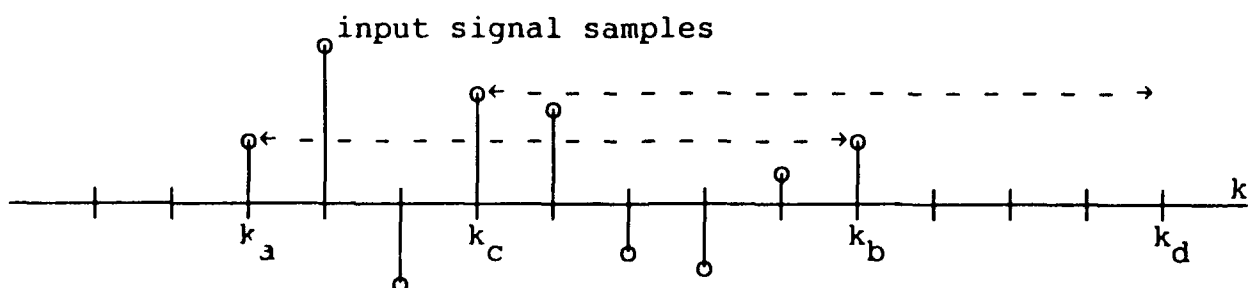


Figure 2. Gated and Observation Intervals; No Noise

## FILTER OUTPUT NOISE CHARACTERIZATION

The filter output noise sequence is obtained from discrete convolution (5) and (1) as

$$y_n(k\Delta) = \Delta \sum_m h(m\Delta) n(k\Delta - m\Delta) \quad \text{for all } k, \quad (8)$$

where it is presumed that the filter  $h(m\Delta)$  has been open to noise input  $n(k\Delta)$  for all time; this results in noise output  $y_n(k\Delta)$  being a zero-mean stationary Gaussian sequence. The filter output noise covariance at general times  $k_1\Delta$  and  $k_2\Delta$  is given by

$$\begin{aligned} C_n(k_1\Delta, k_2\Delta) &= \overline{y_n(k_1\Delta) y_n(k_2\Delta)} = \\ &= \Delta^2 \sum_{mp} h(m\Delta) h(p\Delta) \overline{n(k_1\Delta - m\Delta) n(k_2\Delta - p\Delta)} = \\ &= \Delta^2 \sum_{mp} h(m\Delta) h(p\Delta) R_n(k_1\Delta - k_2\Delta + p\Delta - m\Delta) = \\ &= \sum_j \phi_h(j\Delta) R_n(k_1\Delta - k_2\Delta - j\Delta), \end{aligned} \quad (9)$$

where the filter correlation function is defined as

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta) h(m\Delta - j\Delta) \quad \text{for all } j. \quad (10)$$

The right-hand side of (9) is a function only of the time difference  $k_1\Delta - k_2\Delta$ , in keeping with the stationarity of filter output noise  $y_n(k\Delta)$ .



If the input noise in figure 1 is white, its covariance in (2) becomes

$$C_{n_i}(m\Delta) = \sigma_n^2 \delta_{m0} , \quad (11)$$

where  $\sigma_n^2$  is the input noise power. In this case, the filter output noise covariance in (9) simplifies to

$$C_n(k_1\Delta, k_2\Delta) = \sigma_n^2 \phi_h(k_1\Delta - k_2\Delta) \quad \text{for white input noise} . \quad (12)$$

The numerical evaluation of filter correlation function  $\phi_h(j\Delta)$  in (10), directly from a specified transfer function  $H(f)$ , is considered in appendix B. A possible problem with discontinuous functions is treated in appendix C.

## FILTER OUTPUT SIGNAL CHARACTERIZATION

The filter output signal sequence (when present) follows from (5) and (1) as

$$y_s(k\Delta) = \Delta \sum_m h(k\Delta - m\Delta) s(m\Delta) g(m\Delta) \quad \text{for all } k. \quad (13)$$

The finite duration of gating sequence  $g(k\Delta)$  in (4), as well as the realizability of impulse response  $\Delta h(m\Delta)$ , will serve to terminate the summation in (13) at finite limits.

We presume that filter  $h(m\Delta)$  in figure 1 is realizable; that is,

$$h(m\Delta) = 0 \quad \text{for } m < 0. \quad (14)$$

This makes filter output signal  $y_s(k\Delta)$  in (13) equal to zero for  $k < k_a$ . On the other hand, if the filter has an infinite-duration impulse response  $\Delta h(m\Delta)$ , then  $y_s(k\Delta)$  is nonzero for  $k \geq k_a$ ; see figure 3. However, the filter output noise  $y_n(k\Delta)$  will dominate the signal output for  $k \gg k_b$ , meaning that performance of the detector will be poor in the case when  $k_d \gg k_b$ . That is, too long an observation time in (7), relative to the signal duration, is detrimental to signal detectability.

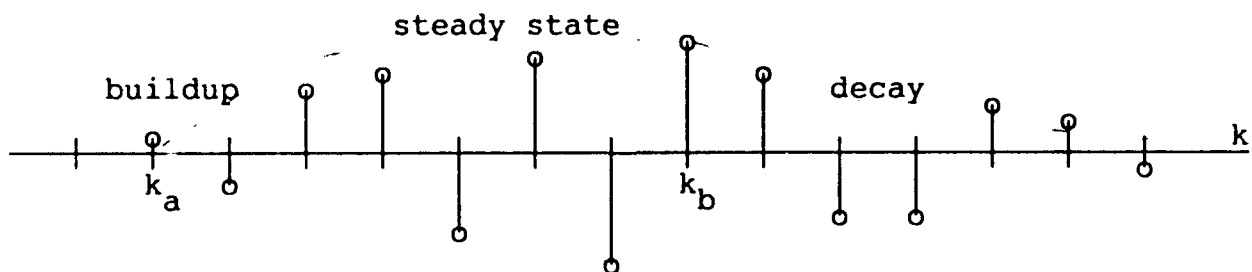


Figure 3. Filter Output Signal Buildup and Decay

The signal output,  $y_s(k\Delta)$  in (13), from filter  $h(m\Delta)$  is not stationary. The filter output signal covariance is defined as

$$C_s(k_1\Delta, k_2\Delta) = \overline{y_s(k_1\Delta) y_s(k_2\Delta)} = C_s(k_2\Delta, k_1\Delta) \quad \text{for all } k_1, k_2. \quad (15)$$

This function is zero for  $k_1 < k_a$  or  $k_2 < k_a$ ; therefore, in the following, we can confine calculation of (15) to  $k_1 \geq k_a$  and  $k_2 \geq k_a$ . The particular filter output signal sample  $y_s(k_a\Delta)$  is nonzero only if  $h(0) \neq 0$ .

Substitution of (13) in (15) and the use of (3) yields

$$\begin{aligned} C_s(k_1\Delta, k_2\Delta) &= \Delta^2 \sum_{mp} h(k_1\Delta - m\Delta) h(k_2\Delta - p\Delta) \overline{s(m\Delta) s(p\Delta)} g(m\Delta) g(p\Delta) = \\ &= \Delta^2 \sum_{mp} h(k_1\Delta - m\Delta) h(k_2\Delta - p\Delta) R_s(m\Delta - p\Delta) g(m\Delta) g(p\Delta). \end{aligned} \quad (16)$$

When we take explicit account of realizability condition (14) and the finite duration of unity gating function  $g(k\Delta)$  in (4), the filter output signal covariance in (16) can be efficiently computed according to

$$C_s(k_1\Delta, k_2\Delta) = \Delta^2 \sum_{m=k_a}^{K_1} h(k_1\Delta - m\Delta) \sum_{p=k_a}^{K_2} h(k_2\Delta - p\Delta) R_s(m\Delta - p\Delta), \quad (17)$$

where

$$K_1 = \min(k_1, k_b), \quad K_2 = \min(k_2, k_b). \quad (18)$$

Since  $k_b \geq k_a$ ,  $k_1 \geq k_a$ ,  $k_2 \geq k_a$ , it follows that  $K_1 \geq k_a$  and  $K_2 \geq k_a$ , thereby making the summations in (17) treat only nonzero entries.

## SUMMER CONSIDERATIONS

The weighted summer in figure 1 and (6) - (7) takes its samples at times  $k\Delta$ , where  $k_c \leq k \leq k_d$ . If  $k_d < k_a$ , then no signal gets into the summer, and the signal at the input cannot be detected at all, whether present or not. Since this is not a useful application of the detector in figure 1, the only situation we will address is that where  $k_d \geq k_a$ , so that some signal (when present) contributes to the summer output.

There are then two possibilities for observation start time  $k_c\Delta$ , as indicated in figure 4 below. If  $k_c < k_a$  (case 1), the summer is taking in noise-only samples for  $k_c \leq k < k_a$ ; this will degrade performance of the detector but is sometimes unavoidable when the signal onset time,  $k_a\Delta$ , is unknown. In this case, we only need to compute signal output covariance  $C_s(k_1\Delta, k_2\Delta)$  for  $k_a \leq k_1, k_2 \leq k_d$ , since  $C_s = 0$  for  $k_1 < k_a$  or  $k_2 < k_a$ .

On the other hand, if  $k_c \geq k_a$  (case 2), signal is taken into the summer on all samples available to it. However, if  $k_c$  is considerably larger than  $k_a$ , a significant portion of the signal contribution can be lost; this degradation of performance is

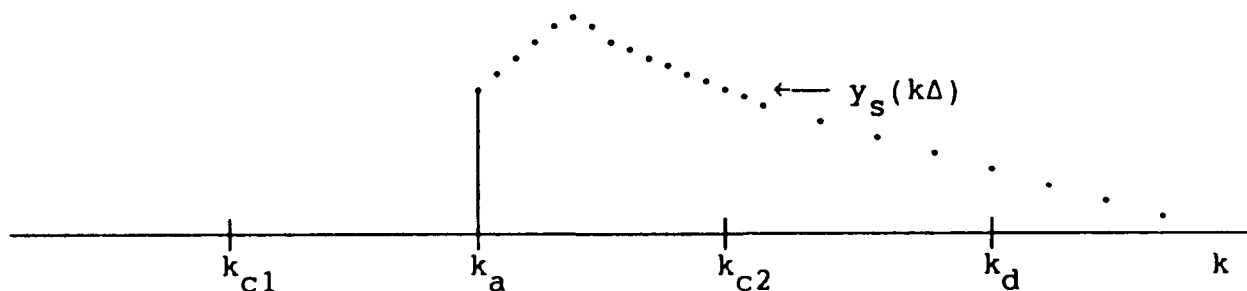


Figure 4. Filter Output Signal  $y_s(k\Delta)$

sometimes unavoidable when  $k_a$  is unknown. In this latter case, we only need to compute  $C_s(k_1\Delta, k_2\Delta)$  for  $k_c \leq k_1, k_2 \leq k_d$ .

The general rule is that we need to compute, from (17), the filter output signal covariance  $C_s(k_1\Delta, k_2\Delta)$  for

$$K_o \equiv \max(k_a, k_c) \leq k_1, k_2 \leq k_d . \quad (19)$$

Then, if  $k_c < k_a$ , use 0 for  $C_s(k_1\Delta, k_2\Delta)$  for those values where  $k_1 < k_a$  or  $k_2 < k_a$ . Of course, noise covariance  $C_n(k_1\Delta, k_2\Delta)$  in (9) or (12) must be computed for  $k_c \leq k_1, k_2 \leq k_d$  in all cases.

Since, from (19),

$$k_1 \geq \max(k_a, k_c) \geq k_a, \quad \text{then} \quad K_1 = \min(k_1, k_b) \geq k_a ; \quad (20)$$

also, since

$$k_2 \geq \max(k_a, k_c) \geq k_a, \quad \text{then} \quad K_2 = \min(k_2, k_b) \geq k_a . \quad (21)$$

Therefore, the sums in (17) always have some entries; that is, the upper limit is never less than the lower limit. The only restriction is

$$k_a \leq k_d . \quad (22)$$

Of course, we must always have  $k_a \leq k_b$  and  $k_c \leq k_d$ .

In summary, the filter output signal covariance follows from (17) and (18), while the filter output noise covariance is available in (9) or (12). The region where the filter output signal covariance  $C_s(k_1\Delta, k_2\Delta)$  calculation is nonzero is crosshatched in figure 5 for the case where  $k_c < k_a$ . If  $k_c \geq k_a$ , then  $C_s(k_1\Delta, k_2\Delta)$  is nonzero in the larger square indicated in figure 5.

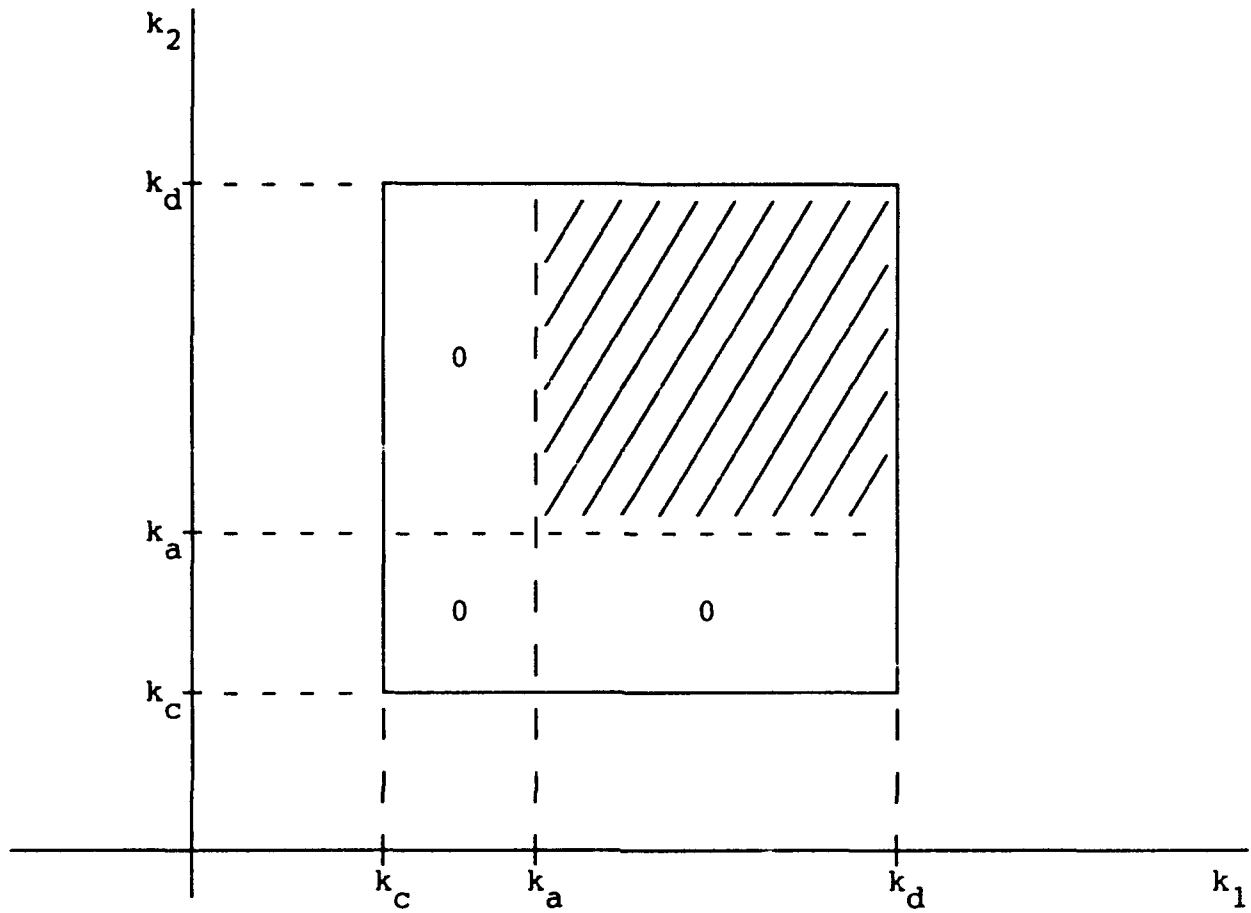


Figure 5. Calculation Region for  $C_s(k_1\Delta, k_2\Delta)$

CHARACTERISTIC FUNCTION OF SYSTEM OUTPUT  $z$ 

The system output  $z$  is given in (6) as a sum of weighted and squared correlated zero-mean Gaussian random variables  $y(k\Delta)$ . Also, the covariance of the noise component  $y_n(k\Delta)$  is given by (9) or (12), while the covariance of the signal component  $y_s(k\Delta)$  (if present) is given by (17) - (18) and figure 5. Therefore, the covariance of  $y(k\Delta)$  is given by

$$C_y(k_1\Delta, k_2\Delta) = C_s(k_1\Delta, k_2\Delta) + C_n(k_1\Delta, k_2\Delta) . \quad (23)$$

The system input signal-to-noise ratio is  $R_s(0)/R_n(0)$  in terms of input covariances (2) and (3).

For nonnegative weights  $\{w(k\Delta)\}$ , we define random variables

$$a_k = \sqrt{w(k\Delta)} y(k\Delta) \quad \text{for } k_c \leq k \leq k_d . \quad (24)$$

Then, (6) - (7) yields the output in the form

$$z = \sum_{k=k_c}^{k_d} a_k^2 , \quad (25)$$

where zero-mean Gaussian random variables  $\{a_k\}$  have covariance

$$C_a(k_1, k_2) = \overline{a_{k_1} a_{k_2}} = \sqrt{w(k_1\Delta) w(k_2\Delta)} C_y(k_1\Delta, k_2\Delta) . \quad (26)$$

In order to find the characteristic function of random variable  $z$  in (25), we consider the square symmetric covariance matrix  $C$  with elements (26) for  $k_c \leq k_1, k_2 \leq k_d$ . Let  $Q$  be the

normalized modal matrix of the orthonormal eigenvectors of  $C$  and let  $\Lambda$  be the diagonal matrix of the eigenvalues  $\{\lambda_k\}$  of  $C$ ; see [1; pages 36 - 39]. Then we have

$$Q^T Q = Q Q^T = I, \quad Q^T C Q = \Lambda. \quad (27)$$

Now let column vector  $A$  be made up of elements  $\{a_k\}$  for  $k_c \leq k \leq k_d$ , as defined in (24), in which case (25) and (26) can be expressed as

$$z = A^T A, \quad C = \overline{A A^T}. \quad (28)$$

Also, let column vector  $B$  be defined by linear transformation

$$B = Q^T A; \quad \text{then} \quad A = Q B. \quad (29)$$

Substitution of (29) into (28) yields

$$z = A^T A = B^T Q^T Q B = B^T B = \sum_{k=k_c}^{k_d} b_k^2, \quad (30)$$

where  $\{b_k\}$  are the elements of  $B$ . At the same time, the covariance matrix of vector  $B$  is

$$\overline{B B^T} = Q^T \overline{A A^T} Q = Q^T C Q = \Lambda, \quad (31)$$

where we used (29), (28), and (27). Thus, the sum for  $z$  in (30) is composed of uncorrelated (and therefore independent) zero-mean Gaussian random variables  $\{b_k\}$ , where the variance of element  $b_k$  is eigenvalue  $\lambda_k$ .

The characteristic function of random variable  $z$  in (30) is,



using the independence and Gaussian character of  $\{b_k\}$ ,

$$\begin{aligned}
 f_z(\xi) &= \overline{\exp(i\xi z)} = \prod_{k=k_c}^{k_d} \overline{\exp(i\xi b_k^2)} = \\
 &= \prod_{k=k_c}^{k_d} \int du \exp(i\xi u^2) (2\pi\lambda_k)^{-1/2} \exp\left(\frac{-u^2}{2\lambda_k}\right) = \\
 &= \prod_{k=k_c}^{k_d} \left(1 - i2\xi\lambda_k\right)^{-1/2}. \tag{32}
 \end{aligned}$$

Although this final result is given as a finite product of  $k_d - k_c + 1$  principal-value square roots, it can be computed as a single principal-value square root of a finite product of complex first-order polynomials, provided that the location of the product in the complex plane is tracked from  $\xi = 0$ ; for example, see [2; pages B-1 and B-2].

## COVARIANCE AND FILTER EXAMPLES

## INPUT SIGNAL CHARACTERIZATION

The input signal covariance will be taken to be a sum of damped exponentials:

$$R_s(\tau) = \sum_{j=1}^J \alpha_j \exp(-|\tau|/\tau_{sj}) \quad \text{for all } \tau, \quad (33)$$

where the  $J$  exponential parameters  $\{\tau_{sj}\}$  are all distinct. The scalings  $\{\alpha_j\}$  and time constants  $\{\tau_{sj}\}$  can be complex, provided that they occur in complex conjugate pairs; that is, input signal covariance  $R_s(\tau)$  must be real for all  $\tau$ . The origin value,

$$R_s(0) = \sum_{j=1}^J \alpha_j = \sigma_s^2, \quad (34)$$

is the input power of stationary signal sequence  $s(k\Delta)$  in (1) and (3), prior to gating. The input signal spectrum corresponding to (33) is, with  $\omega = 2\pi f$ ,

$$G_s(f) = \int d\tau \exp(-i2\pi f\tau) R_s(\tau) = \sum_{j=1}^J \frac{2 \alpha_j \tau_{sj}}{1 + (\omega \tau_{sj})^2}, \quad (35)$$

which is seen to contain  $2J$  distinct poles in the complex  $f$ -plane.  $G_s(f)$  can contain up to  $2J-2$  zeros.

For time sampling increment  $\Delta$ , the sampled input signal covariance is, from (33),

$$R_S(k\Delta) = \sum_{j=1}^J \alpha_j \exp(-a_j |k|) ; \quad a_j = \frac{\Delta}{\tau_{sj}} . \quad (36)$$

The  $J$  exponential parameters  $\{a_j\}$  are distinct, but they can be complex; in fact,  $\{\alpha_j\}$  and  $\{a_j\}$  can both be complex provided that they occur in complex conjugate pairs such that  $R_S(k\Delta)$  is real for all  $k$ . In fact, some of the scalings  $\{\alpha_j\}$  can be negative, provided that total spectrum  $G_S(f)$  in (35) is nonnegative for all  $f$ . The exponential parameters  $\{a_j\}$  in (36) must also satisfy  $\text{Re } a_j > 0$  in order that the covariance tend to zero at  $k = \pm\infty$ .

For a particular pair of complex conjugate terms, say

$$\alpha_1 = \alpha_0 , \quad \alpha_2 = \alpha_0^* , \quad \tau_{s1} = \tau_0 , \quad \tau_{s2} = \tau_0^* , \quad a_1 = a_0 , \quad a_2 = a_0^* , \quad (37)$$

the corresponding part of the (continuous) input signal covariance (33) can be expressed in the alternative form

$$\begin{aligned} R_S^0(\tau) &= \alpha_0 \exp(-|\tau|/\tau_0) + \alpha_0^* \exp(-|\tau|/\tau_0^*) = \\ &= 2 \exp(-\omega_r |\tau|) [\alpha_r \cos(\omega_i |\tau|) + \alpha_i \sin(\omega_i |\tau|)] , \end{aligned} \quad (38)$$

where we have defined the real and imaginary parts

$$\alpha_0 = \alpha_r + i\alpha_i , \quad \frac{1}{\tau_0} = \omega_r + i\omega_i . \quad (39)$$

The sampled signal covariance for component (38) is

$$R_S^0(k\Delta) = 2 \exp(-a_r |k|) [\alpha_r \cos(a_i |k|) + \alpha_i \sin(a_i |k|)] , \quad (40)$$

where  $a_r = \omega_r \Delta$  and  $a_i = \omega_i \Delta$ .

The spectral component associated with component covariance (38) is, by reference to (35) and (39),

$$\begin{aligned}
 G_S^O(f) &= \frac{2 \alpha_o \tau_o}{1 + (\omega \tau_o)^2} + \frac{2 \alpha_o^* \tau_o^*}{1 + (\omega \tau_o^*)^2} = \\
 &= 2 \operatorname{Re} \frac{2 \alpha_o / \tau_o}{\omega^2 + 1/\tau_o^2} = 4 \operatorname{Re} \frac{(\alpha_r + i\alpha_i)(\omega_r + i\omega_i)}{\omega^2 + (\omega_r + i\omega_i)^2} = \\
 &= 4 \frac{\alpha_r \omega_r (\omega_r^2 + \omega_i^2 + \omega^2) + \alpha_i \omega_i (\omega_r^2 + \omega_i^2 - \omega^2)}{(\omega_r^2 + (\omega - \omega_i)^2)(\omega_r^2 + (\omega + \omega_i)^2)} = \\
 &= 2 \frac{\alpha_r \omega_r - \alpha_i(\omega - \omega_i)}{\omega_r^2 + (\omega - \omega_i)^2} + 2 \frac{\alpha_r \omega_r + \alpha_i(\omega + \omega_i)}{\omega_r^2 + (\omega + \omega_i)^2} = \\
 &= \frac{\alpha_i + i\alpha_r}{\omega + \omega_i + i\omega_r} + \frac{\alpha_i - i\alpha_r}{\omega + \omega_i - i\omega_r} + \frac{-\alpha_i + i\alpha_r}{\omega - \omega_i + i\omega_r} + \frac{-\alpha_i - i\alpha_r}{\omega - \omega_i - i\omega_r}. \tag{41}
 \end{aligned}$$

If this spectral component  $G_S^O(f)$  goes negative for some ranges of  $f$ , it must be accompanied by additional terms in summation (35) to keep the total spectrum  $G_S(f)$  nonnegative for all frequencies  $f$ . Spectral component (41) has four poles in the complex  $\omega$ -plane at locations  $\omega = \omega_i \pm i\omega_r$  and  $\omega = -\omega_i \pm i\omega_r$ .

Even if coefficients  $\{\alpha_j\}$  and  $\{\tau_{sj}\}$  in (35) are purely real and the poles are distinct, oscillatory behavior of spectrum  $G_S(f)$  is still attainable. For example, with

$$J = 3, \quad \{a_j\} = 1, 8, 15, \quad \{\alpha_j\} = 11, -48, 75, \tag{42}$$

the spectrum (35) is displayed versus  $\omega\Delta$  in figure 6. Actually plotted is the scaled spectrum

$$\frac{G_s(f)}{2\Delta} = \sum_{j=1}^J \frac{\alpha_j a_j}{(\omega\Delta)^2 + a_j^2}, \quad (43)$$

where we have used (36) to eliminate  $\{\tau_{sj}\}$ . The negative coefficient for  $\alpha_2$  causes the dip near  $\omega\Delta = 4$ .

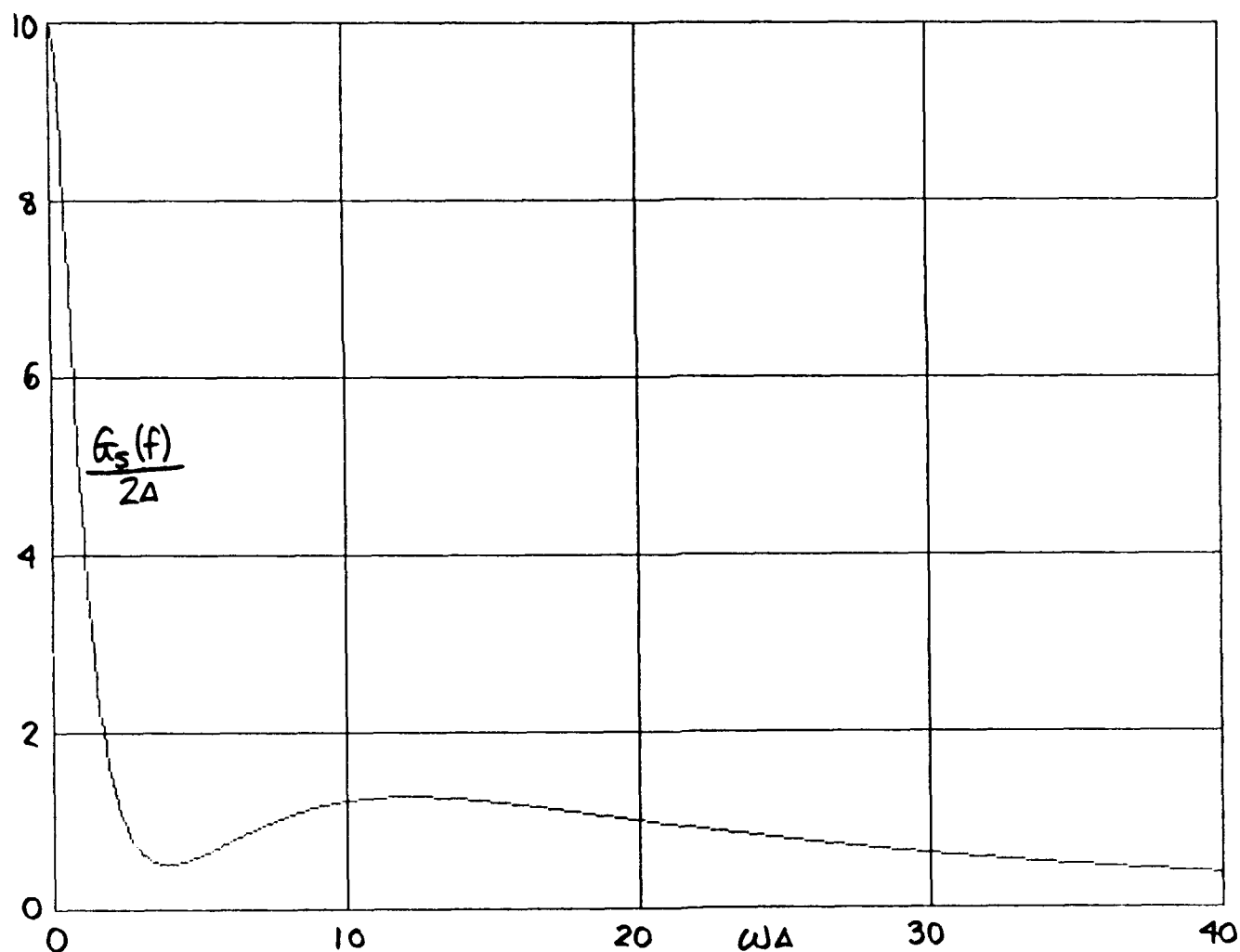


Figure 6. Spectrum (43) for Parameters (42)

## INPUT NOISE CHARACTERIZATION

The presentation in this subsection exactly parallels that above for the input signal. The input noise covariance is again taken to be a sum of damped exponentials of the form

$$R_n(\tau) = \sum_{m=1}^M \beta_m \exp(-|\tau|/\tau_{nm}) \quad \text{for all } \tau, \quad (44)$$

where the  $M$  (complex) noise time constants  $\{\tau_{nm}\}$  are all distinct from one another. (Some of them may equal some of the signal time constants  $\{\tau_{sj}\}$  in (33), if desired.) The origin value,

$$R_n(0) = \sum_{m=1}^M \beta_m = \sigma_n^2, \quad (45)$$

is the input power of noise sequence  $n(k\Delta)$  in (1).

For time sampling increment  $\Delta$ , the sampled input noise covariance is, from (44),

$$R_n(k\Delta) = \sum_{m=1}^M \beta_m \exp(-b_m |k|); \quad b_m = \frac{\Delta}{\tau_{nm}}. \quad (46)$$

The noise exponential parameters  $\{b_m\}$  are distinct, but they can be complex; however,  $\text{Re } b_m > 0$ . (Some of the  $\{b_m\}$  can equal some of the  $\{a_j\}$  in (36).) The special case of white noise,

$R_n(k\Delta) = \sigma_n^2 \delta_{ko}$ , is handled by choosing  $M = 1$ ,  $\beta_1 = \sigma_n^2$ ,  $\tau_{n1} \rightarrow 0$ ,  $b_1 \rightarrow +\infty$ .

## FILTER CHARACTERIZATION

We first consider an analog filter with a rational transfer function with  $N$  distinct poles, namely

$$H(f) = \sum_{n=1}^N \frac{\psi_n}{1 + i2\pi f \tau_{hn}} , \quad (47)$$

where  $\{\tau_{hn}\}$  are distinct (complex) time constants.  $H(f)$  can have up to  $N-1$  zeros. The corresponding impulse response is

$$h(\tau) = \begin{cases} \sum_{n=1}^N \frac{\psi_n}{\tau_{hn}} \exp\left(\frac{-\tau}{\tau_{hn}}\right) & \text{for } \tau \geq 0 \\ 0 & \text{for } \tau < 0 \end{cases} , \quad (48)$$

which is required to be real for all  $\tau$ . The sampled impulse response is then of the form

$$\Delta h(m\Delta) = \sum_{n=1}^N \psi_n c_n \exp(-c_n m) \quad \text{for } m \geq 0 , \quad c_n = \frac{\Delta}{\tau_{hn}} , \quad (49)$$

where the  $N$  filter exponential parameters  $\{c_n\}$  are distinct from one another; also,  $\text{Re } c_n > 0$ .

The filter correlation function is available by substitution of (49) in (10), with the result

$$\phi_h(j\Delta) = \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |j|) \quad \text{for all } j , \quad (50)$$

where

$$v_n = \sum_{k=1}^N \frac{\psi_k c_k}{1 - \exp(-c_n - c_k)} \quad \text{for } 1 \leq n \leq N . \quad (51)$$

The form of filter correlation  $\phi_h(j\Delta)$  in (50) is identical to the sampled input signal covariance (36), in that (50) is also composed of  $N$  distinct exponentials; hence, the comments and examples given there are relevant here also.

A useful alternative exists to the calculation of constants  $\{v_n\}$  via (51). Namely, define  $z$  transform filter

$$\underline{H}(z) = \sum_{m=0}^{\infty} z^{-m} \Delta h(m\Delta) . \quad (52)$$

Replace  $n$  by  $k$  in (49) and substitute the result into (52); then an interchange of summations yields

$$\underline{H}(z) = \sum_{k=1}^N \frac{\psi_k c_k}{1 - \frac{1}{z} \exp(-c_k)} . \quad (53)$$

It then follows immediately by comparison with (51) that

$$v_n = \underline{H}(\exp(c_n)) \quad \text{for } 1 \leq n \leq N . \quad (54)$$

When  $\underline{H}(z)$  is available as a rational function, (54) can be used directly, instead of summation (51).

The exponential parameters  $\{c_n\}$ , in impulse response (49) and corresponding  $N$ -pole filter (53), can be complex, provided that they occur in complex conjugate pairs and that their corresponding coefficients  $\{\psi_n\}$  are also conjugate pairs. That is, the sum of two conjugate terms of the form of (49) must be real for all  $m \geq 0$ . This generality allows for oscillatory impulse responses such as yielded by narrowband filters.



## FILTER OUTPUT NOISE COVARIANCE

In the sequel, we will find use for the following closed form for the discrete convolution of two exponentials:

$$\begin{aligned} \exp[-c|k|] \otimes \exp[-b|k|] &\equiv \sum_j \exp[-c|j| - b|k-j|] = \\ &= \left\{ \begin{array}{ll} \frac{\sinh(c) \exp(-b|k|) - \sinh(b) \exp(-c|k|)}{\cosh(c) - \cosh(b)} & \text{for } b \neq c \\ \left[ |k| + \frac{\cosh(c)}{\sinh(c)} \right] \exp(-c|k|) & \text{for } b = c \end{array} \right\}. \quad (55) \end{aligned}$$

The filter output noise covariance is given by (9). If we define

$$C(k\Delta) = \sum_j \phi_h(j\Delta) R_n(k\Delta - j\Delta), \quad (56)$$

then the stationarity of filter output noise  $y_n(k\Delta)$  in (8) allows us to express (9) as

$$C_n(k_1\Delta, k_2\Delta) = C(k_1\Delta - k_2\Delta). \quad (57)$$

Therefore, we can concentrate on evaluation of  $C(k\Delta)$  in (56) for representative filter correlations  $\phi_h$  and input noise covariances  $R_n$ .

In particular, we will use the general filter correlation (50) and input noise covariance (46), where we presume that none of the filter parameters  $\{c_n\}$  are equal to any of the input noise parameters  $\{b_m\}$ . That is,

$$c_n \neq b_m \quad \text{for all } n, m. \quad (58)$$

Substitution of (50) and (46) in (56) and use of (55) results in

$$\begin{aligned} C(k\Delta) &= \sum_j \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |j|) \sum_{m=1}^M \beta_m \exp(-b_m |k-j|) = \\ &= \sum_{n=1}^N \sum_{m=1}^M \psi_n c_n v_n \beta_m \sum_j \exp[-c_n |j| - b_m |k-j|] = \\ &= \sum_{m=1}^M \mu_m \exp(-b_m |k|) + \sum_{n=1}^N \gamma_n \exp(-c_n |k|), \end{aligned} \quad (59)$$

where we defined auxiliary constants

$$\begin{aligned} \mu_m &= \beta_m \sum_{n=1}^N \frac{\psi_n c_n v_n \sinh(c_n)}{\cosh(c_n) - \cosh(b_m)} \quad \text{for } 1 \leq m \leq M, \\ \gamma_n &= \psi_n c_n v_n \sum_{m=1}^M \frac{\beta_m \sinh(b_m)}{\cosh(b_m) - \cosh(c_n)} \quad \text{for } 1 \leq n \leq N. \end{aligned} \quad (60)$$

Condition (58) keeps all the denominators in (60) from becoming zero. On the other hand, if one of the filter parameters  $\{c_n\}$  is equal to one of the noise parameters  $\{b_m\}$ , then it is merely necessary to utilize instead the second line of (55) for that particular  $n, m$  pair in the  $j$  sum in the middle line of (59).

End result (59) for the filter output noise covariance is a compact expression that is capable of being quickly computed, once constants  $\{\mu_m\}$  and  $\{\gamma_n\}$  have been evaluated by means of (60) and stored. Again, we encounter a sum of decaying exponentials, albeit of  $M + N$  terms now.

## FILTER OUTPUT NOISE COVARIANCE FOR WHITE NOISE INPUT

When the input noise is white, then as noted under (46), we have for the input noise covariance,

$$R_n(k\Delta) = \sigma_n^2 \delta_{k0}, \quad \text{that is, } M = 1, \beta_1 = \sigma_n^2, b_1 \rightarrow +\infty. \quad (61)$$

Use of this result in (60) yields  $\mu_1 \rightarrow 0$ ,  $\gamma_n \rightarrow \sigma_n^2 \psi_n c_n v_n$ , in which case the filter output noise covariance (59) becomes

$$C(k\Delta) = \sigma_n^2 \sum_{n=1}^N \psi_n c_n v_n \exp(-c_n |k|), \quad (62)$$

where  $\{v_n\}$  are given by (51).

Actually, (62) is a special case of the general white noise result obtained by substitution of (61) into (56); namely, for arbitrary filter  $\Delta h(m\Delta)$ , the filter output noise covariance is

$$C(k\Delta) = \sigma_n^2 \phi_h(k\Delta) \quad \text{for white noise input,} \quad (63)$$

where filter correlation  $\phi_h$  is given by (10).

## FILTER OUTPUT SIGNAL COVARIANCE

The nonstationary filter output signal covariance is given by double summation (17) in conjunction with (18). If we substitute input signal covariance  $R_s(k\Delta)$  given by (36), along with filter impulse response  $\Delta h(m\Delta)$  given by (49), into (17), there follows

$$\begin{aligned}
 C_s(k_1\Delta, k_2\Delta) &= \sum_{m=k_a}^{K_1} \sum_{n=1}^N \psi_n c_n \exp(-c_n(k_1-m)) \sum_{p=k_a}^{K_2} \sum_{q=1}^N \psi_q c_q \times \\
 &\quad \times \exp(-c_q(k_2-p)) \sum_{j=1}^J \alpha_j \exp(-a_j|m-p|) = \\
 &= \sum_{j=1}^J \alpha_j \sum_{n=1}^N \psi_n c_n \exp(-c_n k_1) \sum_{q=1}^N \psi_q c_q \exp(-c_q k_2) \times \\
 &\quad \times \sum_{m=k_a}^{K_1} \exp(c_n m) \sum_{p=k_a}^{K_2} \exp(c_q p - a_j|m-p|) . \quad (64)
 \end{aligned}$$

In order to evaluate the innermost double summation in (64), we let  $A = \exp(a_j)$ ,  $B = \exp(c_q)$ ,  $C = \exp(c_n)$  and we presume that  $k_1 \leq k_2$ . It then follows from (18) that  $K_1 \leq K_2$ , and the double sum in the last line of (64) can be developed thusly:

$$\begin{aligned}
 S(A, B, C, k_a, K_1, K_2) &\equiv \sum_{m=k_a}^{K_1} C^m \sum_{p=k_a}^{K_2} B^p A^{-|m-p|} = \quad (65) \\
 &= \sum_{m=k_a}^{K_1} C^m \left[ \sum_{p=k_a}^m B^p A^{p-m} + \sum_{p=m}^{K_2} B^p A^{m-p} - B^m \right] =
 \end{aligned}$$

$$= \sum_{m=k_a}^{K_1} (C/A)^m \sum_{p=k_a}^m (BA)^p + \sum_{m=k_a}^{K_1} (CA)^m \sum_{p=m}^{K_2} (B/A)^p - \sum_{m=k_a}^{K_1} (CB)^m. \quad (66)$$

Now we use the fact that

$$\sum_{m=L}^M z^m = \frac{z^L - z^{M+1}}{1 - z} \quad \text{for } z \neq 1, \quad L \leq M, \quad (67)$$

and we define auxiliary function

$$g(z, k) \equiv \frac{z^k}{1 - z} \quad \text{for } z \neq 1. \quad (68)$$

Then, after some amount of manipulation, the sums in (66) can be expressed in the closed form

$$\begin{aligned} S(A, B, C, k_a, K_1, K_2) = & g(CB, k_a) \frac{A^2 - CB}{(A-B)(A-C)} + g(CB, K_1+1) \frac{B(A^2 - 1)}{(A-B)(1-BA)} \\ & + g(B/A, K_2+1) \left[ g(CA, K_1+1) - g(CA, k_a) \right] - g(BA, k_a) g(C/A, K_1+1), \end{aligned} \quad (69)$$

provided that  $B \neq A$  and  $C \neq A$ . (B cannot equal  $1/A$ , nor can C equal  $1/A$  or  $1/B$ , because the real parts of  $c_n$  and  $a_j$  are always positive.)

We now employ definition (65) in order to express the filter output signal covariance in (64) in the final form

$$\begin{aligned} C_s(k_1 \Delta, k_2 \Delta) = & \sum_{j=1}^J \alpha_j \sum_{n=1}^N \psi_n c_n \exp(-c_n k_1) \sum_{q=1}^N \psi_q c_q \exp(-c_q k_2) \times \\ & \times S(\exp(a_j), \exp(c_q), \exp(c_n), k_a, K_1, K_2), \end{aligned} \quad (70)$$

provided that none of the filter parameters  $\{c_n\}$  are equal to any of the input signal parameters  $\{a_j\}$ . That is,

$$c_n \neq a_j \quad \text{for all } n, j. \quad (71)$$

This result, (70), cannot be further reduced without the various parameters being specified. The triple summation will not be overly time consuming unless the input signal or filter are characterized by a large number of poles, that is, large  $J$  or  $N$ .

The result in (70) holds for  $k_1 \leq k_2$ ; the range for  $k_1 > k_2$  is most easily handled by use of symmetry relation (15). Also, a significant computational aid is available by using the recurrence relation for the  $g(z, k)$  function in (68), namely  $g(z, k) = z g(z, k-1)$ .

In summary,  $k_a, k_b, k_c, k_d$  are given integers satisfying  $k_a \leq k_b, k_c \leq k_d, k_a \leq k_d$ . Also,  $K_0 = \max(k_a, k_c)$ ,  $K_1 = \min(k_1, k_b)$ ,  $K_2 = \min(k_2, k_b)$ . Since we keep  $k_1 \leq k_2$ , then  $K_1 \leq K_2$ . Integers  $k_1$  and  $k_2$  must vary over the range

$$K_0 \leq k_1 \leq k_2 \leq k_d, \quad (72)$$

meaning that  $K_1$  varies in the range  $\min(K_0, k_b)$  to  $\min(k_d, k_b)$ .

From (69), we see that we have to compute function values  $g(CB, k_a), g(CA, k_a), g(BA, k_a)$ , as well as the arrays of function values of  $g(CB, k), g(CA, k), g(B/A, k), g(C/A, k)$  for  $k$  in the range  $\min(K_0, k_b)+1$  to  $\min(k_d, k_b)+1$ .

## PROGRAMS

A program listing for the white noise input case is presented in appendix D; it consists of two separate parts. The first part computes the filter output signal and noise covariance matrices for unity input signal and noise powers, the total filter output covariance matrix of elements  $C_y$  in (23) for the various signal-to-noise ratios of interest, the weighted covariance matrix  $C$  of elements  $C_a$  in (26) and its corresponding eigenvalue matrix  $\Lambda$  in (27). The output of these eigenvalues to storage completes the first part.

The second part lists the program that takes in these eigenvalues and computes the exceedance distribution functions, for noise-alone as well as for signal present, from characteristic function (32); these are the false alarm and detection probabilities, respectively. The precise method of handling all the widely different signal-to-noise ratios of interest is presented in the next section. At the end of appendix D, there is also a listing of the simulation program utilized to verify these theoretical derivations and results.

The corresponding program for the colored noise case is listed in appendix E. A similar breakdown into two parts has been adopted; however, once the eigenvalues have been computed at the end of the first part, the identical program in part 2 of appendix D is used for distribution calculations and is therefore not listed again. Also, the simulation program use to check the colored noise case is similar to that in part 3 of appendix D.

## ON CALCULATION OF RECEIVER OPERATING CHARACTERISTICS

The evaluation of the receiver operating characteristics for a number of different input signal-to-noise ratios typically involves the Fourier transforms of a set of characteristic functions of the decision variable, which frequently have widely different ranges and rates of variation. In order to make these calculations tractable and easily plotted, it is necessary to resort to FFTs [2] with carefully chosen sizes and sampling increments. We now present the reasoning behind our choices of these parameters and their interrelationships.

For ease of presentation, we will consider the numerical evaluation of a probability density function  $p_0(u)$  from a given (noise-only) characteristic function  $f_0(\xi)$ ; the extension to an exceedance distribution function [2] is immediate. We have

$$\begin{aligned} p_0(u) &= \frac{1}{2\pi} \int d\xi \exp(-i\xi u) f_0(\xi) = \\ &\approx \frac{1}{2\pi} \Delta_0 \sum_k \exp(-ik\Delta_0 u) f_0(k\Delta_0) \equiv \tilde{p}_0(u) , \end{aligned} \quad (73)$$

where the trapezoidal approximation with sampling increment  $\Delta_0$  in  $\xi$  has been employed. The approximation  $\tilde{p}_0(u)$  defined by (73) is an aliased version of  $p_0(u)$  and has period  $u_0 = 2\pi/\Delta_0$  in  $u$ . In order to keep aliasing effects negligible in  $\tilde{p}_0(u)$ , it will be necessary to choose  $\xi$  increment  $\Delta_0$  small enough that the aliasing lobes at separation  $u_0$  don't overlap. Therefore, we can restrict the evaluation of  $\tilde{p}_0(u)$  to  $N_0$  equally spaced points over that interval  $u_0$ , according to



$$\tilde{p}_0\left(\frac{2\pi n}{N_0\Delta_0}\right) = \frac{1}{2\pi} \Delta_0 \sum_k \exp(-i2\pi kn/N_0) f_0(k\Delta_0) \quad \text{for } 0 \leq n \leq N_0-1 . \quad (74)$$

Furthermore, by collapsing the sequence  $\{f_0(k\Delta_0)\}$  modulo  $N_0$ , (74) can be accomplished exactly and efficiently as an  $N_0$ -point FFT when  $N_0$  is a power of 2; see [2]. The increment in argument  $u$  of  $\tilde{p}_0(u)$  in (74) and the maximum useful value of  $u$  are

$$\delta_0 = \frac{2\pi}{N_0\Delta_0} , \quad u_0 = \frac{2\pi}{\Delta_0} . \quad (75)$$

Now suppose that we also want to evaluate the probability density function  $p_1(u)$  corresponding to a different (signal present) characteristic function  $f_1(\xi)$  according to

$$\begin{aligned} p_1(u) &= \frac{1}{2\pi} \int d\xi \exp(-i\xi u) f_1(\xi) = \\ &\approx \frac{1}{2\pi} \Delta_1 \sum_k \exp(-ik\Delta_1 u) f_1(k\Delta_1) \equiv \tilde{p}_1(u) , \end{aligned} \quad (76)$$

where the sampling increment in  $\xi$  is now  $\Delta_1$ . By identical reasoning to that above, we can get samples of periodic (aliased) approximation  $\tilde{p}_1(u)$  according to

$$\tilde{p}_1\left(\frac{2\pi n}{N_1\Delta_1}\right) = \frac{1}{2\pi} \Delta_1 \sum_k \exp(-i2\pi kn/N_1) f_1(k\Delta_1) \quad \text{for } 0 \leq n \leq N_1-1 , \quad (77)$$

which can be realized as an  $N_1$ -point FFT. The increment in  $u$  and the maximum useful value of  $u$  for this latter case are

$$\delta_1 = \frac{2\pi}{N_1\Delta_1} , \quad u_1 = \frac{2\pi}{\Delta_1} . \quad (78)$$

If we now want to eliminate the arguments of functions  $\tilde{p}_0(u)$  and  $\tilde{p}_1(u)$  and be able to directly plot  $\tilde{p}_1(u)$  versus  $\tilde{p}_0(u)$  (that is,  $y$  versus  $x$ ), this can be easily accomplished if we force the  $u$  increments in (74) and (77) to be equal, namely,  $\delta_1 = \delta_0$ . But, then, according to (75) and (78), this means choosing

$$N_1 \Delta_1 = N_0 \Delta_0 . \quad (79)$$

Since integers  $N_0$  and  $N_1$  will be limited to powers of 2, this will mean repeated halving of the  $\xi$  increments  $\Delta_0$  and  $\Delta_1$  in (73) and (76), respectively. The values of  $\tilde{p}_1$  yielded by (77) can then be directly plotted versus those values of  $\tilde{p}_0$  yielded by (74), up to  $n = \min(N_0, N_1) = N_0$ .

There are several alternative procedures that could have been adopted. For example, one option is to halve the  $\xi$  increment, that is, take  $\Delta_1 = \Delta_0/2$ , but keep  $N_1 = N_0$ . This will require discarding every other value put out by FFT (74), in order that the remaining values occur at the same  $u$  arguments yielded by (77). This is wasteful of FFT (74) and has not been adopted here. Nor have we employed the possibility of interpolation of (77) to determine approximately what the values of  $\tilde{p}_1$  would be at the arguments of  $\tilde{p}_0$  in (74).

The need to decrease the size of the  $\xi$  increment from  $\Delta_0$  in (74) to  $\Delta_1$  in (77) is fundamental. It arises from the fact that as the input signal-to-noise ratio is increased, the contribution of the corresponding probability density function  $p_k(u)$ ,  $k = 0, 1, 2, \dots$ , moves to higher values on the  $u$  scale (thereby

leading to the desired higher detection probabilities). In order that approximation  $\tilde{p}_k(u)$  not be severely aliased, the  $\xi$  increment  $\Delta_k$  must therefore be decreased; for example, see upper limits  $u_0$  and  $u_1$  in (75) and (78), respectively.

With these points in mind, the following procedure has been adopted and utilized in the programs in this report. First, for the noise-only probability density function  $p_0(u)$ , a satisfactory value for  $\xi$ -increment  $\Delta_0$  in (73) is found such that aliasing is insignificant in  $\tilde{p}_0(u)$ . This selection of  $\Delta_0$  is arrived at by looking at a plot of (74) and modifying  $\Delta_0$  appropriately by trial and error. An FFT size of  $N_0 = 128$  is utilized to evaluate (74), which is then stored; this size for  $N_0$  has generally proven to be sufficient to keep track of the variations of  $p_0(u)$ .

When we encounter the characteristic function  $f_1(\xi)$  and probability density function  $p_1(u)$  for the first (lowest) nonzero signal-to-noise ratio of interest, it is usually necessary to decrease the  $\xi$  increment to  $\Delta_1 = \Delta_0/2$  or  $\Delta_0/4$  in order to control the aliasing inherent in approximation  $\tilde{p}_1(u)$ . At the same time,  $N_1$  is scaled up by 2 or 4, according to (79), thereby maintaining the same  $u$  arguments for (74) and (77), as desired. Again, a plot of aliased version  $\tilde{p}_1(u)$ , obtained by means of (77), serves as the decision point for making an acceptable choice of value for  $\Delta_1$ .

For the successively larger signal-to-noise ratios and their corresponding probability density functions  $p_k(u)$ ,  $k=1,2,\dots$ , occasional repeated halvings of the  $\xi$  increment according to

$\Delta_k = \Delta_{k-1}/2$  are made, but only when needed to keep aliasing insignificant in  $\tilde{p}_k(u)$ . At those times that such a halving is necessary and made, the FFT size is doubled according to  $N_k = 2 N_{k-1}$ . Otherwise,  $\Delta_k$  and  $N_k$  are kept at their same values as used for the  $k-1$  run. By the time the largest signal-to-noise ratio case of interest is reached, the FFT size  $N_k$  can get rather large. We are limited in our application to a maximum FFT size of 16384; if aliasing is significant at this size, we are unable to accurately numerically handle these larger signal-to-noise ratios by the adopted procedure. Interpolation would then be the recommended alternative, as alluded to above.

The procedure above has been explained in terms of probability density functions rather than the exceedance distribution functions which are of actual interest. That is, the relevant exceedance distribution functions are plots of the detection probabilities for various signal-to-noise ratios versus the false alarm probability. The only basic change is to replace the top lines of (73) and (76) by the exceedance distributions, which are also obtained by means of Fourier transforms, namely [2; (5) - (6)]

$$E_k(u) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi \operatorname{Im} \left\{ \exp(-i\xi u) \frac{f_k(\xi)}{\pi \xi} \right\}, \quad (80)$$

for  $k = 0, 1, 2, \dots$

## RESULTS

## COMPARISON WITH SIMULATION

In figure 7, the cumulative (CDF) and exceedance distribution functions (EDF) for a white noise input excitation are presented. The input signal and filter are both characterized by one pole, that is,  $J = 1$  in (33) - (36) and  $N = 1$  in (47) - (50). The simulation employed  $1E6$  trials and verifies the theoretical calculations down to the probability level of  $1E-5$  plotted. A listing of the simulation program, including all the parameter values that were utilized here, is given at the end of appendix D. The gating and observation parameters are  $k_a = 4$ ,  $k_b = 11$  and  $k_c = 2$ ,  $k_d = 16$ , respectively; thus, it can be seen that the transient buildup and decay are a prominent part of the filter output signal for this particular example.

The results in figure 8 pertain to a colored noise input with one pole, that is,  $M = 1$  in (44) - (46). Again,  $1E6$  trials were used in the simulation and they confirm the theoretical result down to the  $1E-5$  level of probabilities. A listing of the simulation program is given at the end of appendix D.

Absolute time is unimportant to the performance of the energy detector in figure 1. That is, any common constant can be added to or subtracted from  $k_a$ ,  $k_b$ ,  $k_c$ ,  $k_d$ , without changing the receiver operating characteristics. Thus,  $k_a$  could always be selected as 0 without loss of generality.

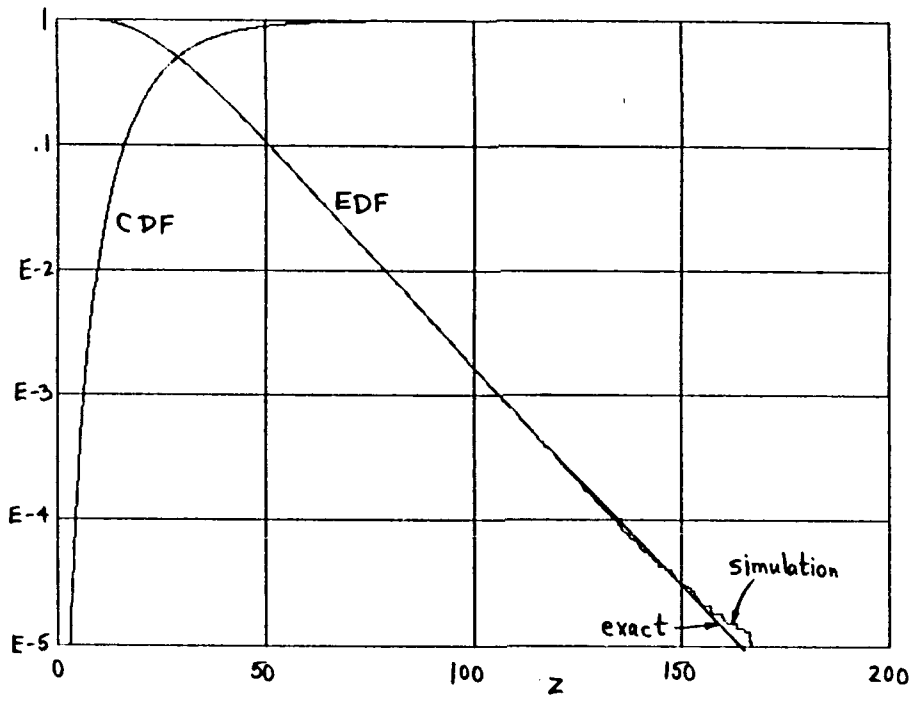


Figure 7. Cumulative and Exceedance Distributions for White Noise

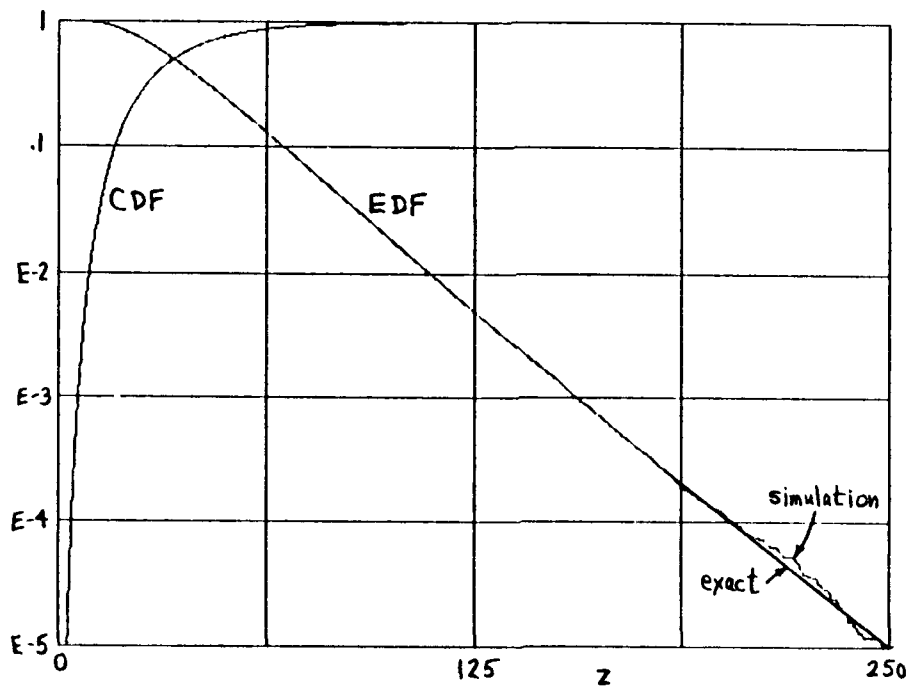


Figure 8. Cumulative & Exceedance Distributions for Colored Noise

## RECEIVER OPERATING CHARACTERISTICS

The receiver operating characteristics, that is, detection probability  $P_D$  versus false alarm probability  $P_F$  for a set of signal-to-noise ratios, for the system in figure 1 will be considered in this section. The first example is evaluated for the following set of parameters, where all times are in seconds. Also, all parameters are real.

$$\begin{aligned} k_a &= 4, k_b = 11, k_c = 2, k_d = 16, \Delta = .2, \\ J &= 3, \{\alpha_j\} = \{11 \quad -48 \quad 75\}, \{\tau_{sj}\} = \{1 \quad 1/8 \quad 1/15\}, \\ M &= 2, \{\beta_m\} = \{39 \quad 60\}, \{\tau_{nm}\} = \{.2 \quad .4\}, \\ N &= 4, \{\psi_n\} = \{1 \quad 2 \quad 3 \quad 4\}, \{\tau_{hn}\} = \{.3 \quad .5 \quad .7 \quad .9\}. \end{aligned} \quad (81)$$

Twenty nonzero values of the system input signal-to-noise ratio

$$R = \frac{\sigma_s^2}{\sigma_n^2} \quad (82)$$

were utilized, namely  $R = 5(1.2)27.8$  dB. For computation of the exceedance distribution function  $P_F$  directly from the characteristic function, the initial  $\xi$  increment  $\Delta_0$  in (73) and (74) was taken as .00025; this yielded round-off errors less than  $1E-15$  in the false alarm probability calculation. Repeated occasional halving of the  $\xi$  increment, as explained in the previous section, was utilized, eventually requiring an FFT size of 16384 for the largest signal-to-noise ratios cases for  $P_D$ .

The receiver operating characteristics are plotted on normal probability paper in figure 9 and cover a wide range of values,

ranging from  $1E-10$  for  $P_F$  to  $.999$  for  $P_D$ . Since Gaussian random variables would plot on such paper as a set of parallel straight lines, the curvatures of these results illustrate that the Gaussian assumption for decision variable  $z$  in figure 1 is not warranted, at least for this example. The major reason for this behavior is the relatively small number of samples used in the detector; see the top line of (81). This is also a partial explanation for the rather large values of the per-sample signal-to-noise ratio  $R$  required at the high quality region at the top left of figure 9. Another factor to notice is that the input signal duration is only  $k_b\Delta - k_a\Delta = 1.4$  seconds, meaning that the filter output signal never reaches steady state before the input signal is turned off; all these transient signal effects and their limitations on performance have been included exactly in the analysis and numerical results presented here.

The effect of halving the sampling increment  $\Delta$  in figure 1 is considered in the next example but done in such a manner as to keep the total input signal duration the same. That is, the parameter values in (81) are kept the same except for the following changes:

$$k_a = 8, k_b = 22, k_c = 4, k_d = 32, \Delta = .1. \quad (83)$$

Notice that all absolute times, such as  $k_a\Delta$ , are kept fixed. The corresponding receiver operating characteristics are plotted in figure 10. Performance has been degraded by a couple of dB. For example, to realize  $P_F = 1E-3$  and  $P_D = .5$ , the per-sample input



signal-to-noise ratio  $R$  must now be 16 dB, whereas it was 13.8 dB in the previous figure. This is a degradation of 2.2 dB. The main reason for this behavior is again the inability of the filter output signal to reach steady state and thereby contribute significantly to the summer output. By contrast, the filter output noise is in steady state (by assumption) for both examples.

We now return to the original sampling increment  $\Delta = .2$  seconds employed in (81). When the observation interval coincides with the input signal nonzero-excitation duration, that is,  $k_a = k_c = 4$ ,  $k_b = k_d = 11$ , the receiver operating characteristics in figure 11 illustrate additional improvement. For example, probabilities  $P_F = 1E-3$  and  $P_D = .5$  can now be realized with  $R = 13.1$  dB, which is an improvement of .7 dB relative to the signal-to-noise ratio required for the broader observation interval,  $k_c = 2$ ,  $k_d = 16$ , used in (81).

Finally, an example with a wide observation interval, namely  $k_c = 0$ ,  $k_d = 25$ , is displayed in figure 12. The desired probabilities can now be achieved only if the input signal-to-noise ratio is increased to 14.7 dB, a degradation of 1.6 dB relative to the best case above.

These examples illustrate the utility of being able to investigate quantitatively and accurately the effects of nonstationarity in a system, without having to make questionable assumptions about, for example, how closely steady state was or was not realized. They also afford a dependable verification or

rejection of the Gaussian approximation for the decision variable; in a related work [3], the latter approximation was found to be too optimistic for most working ranges of this detection system.

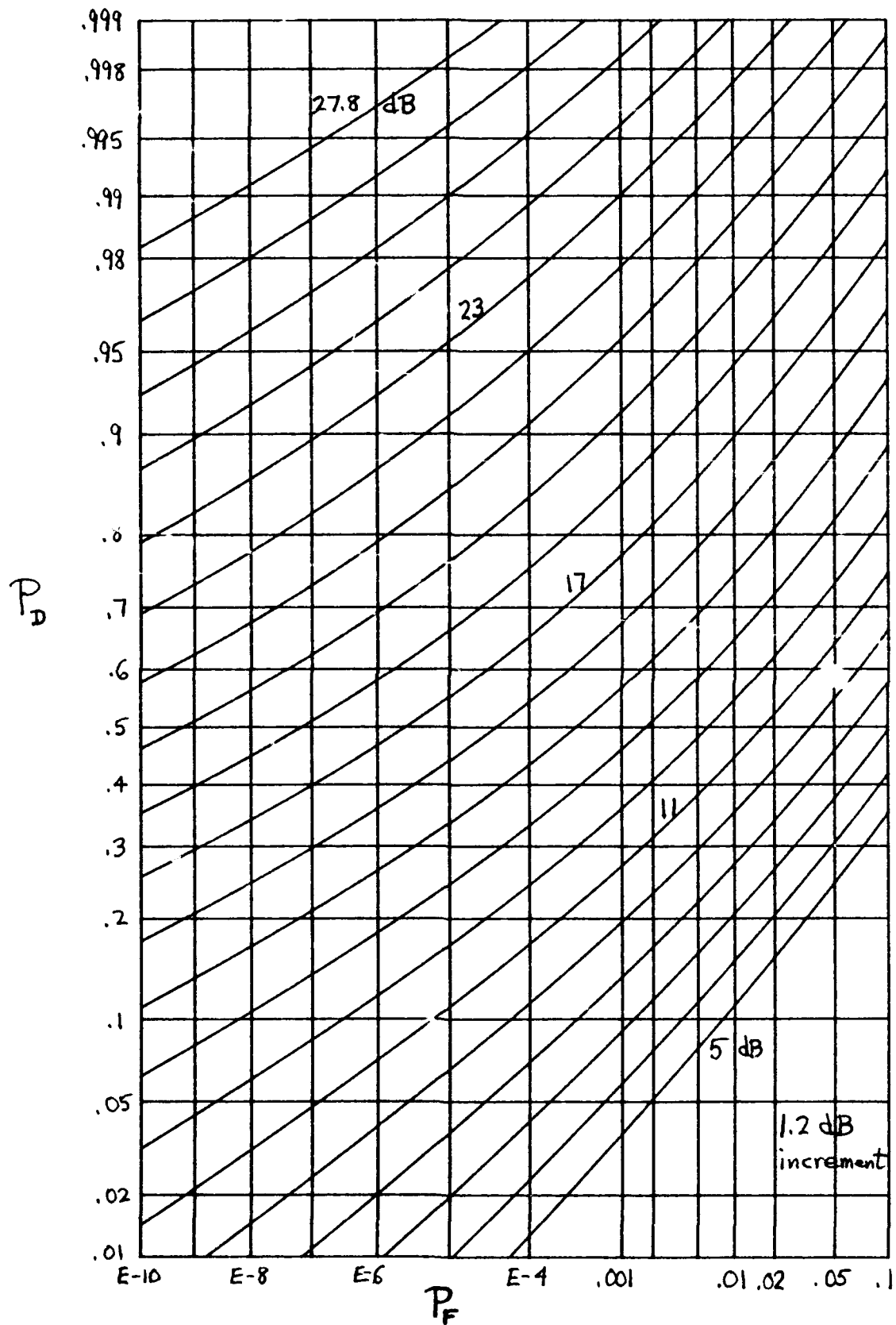


Figure 9. Receiver Operating Characteristics; Example A

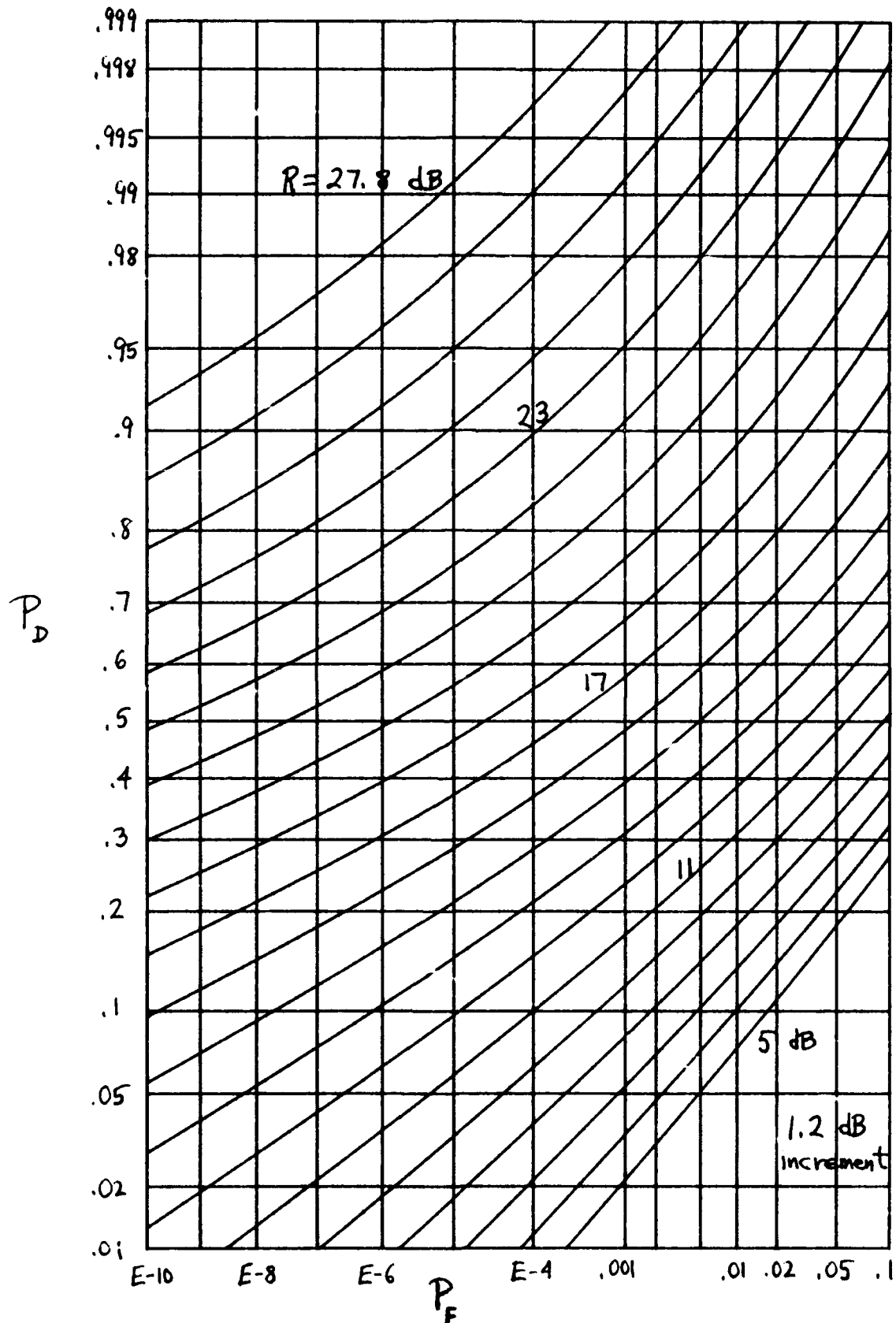


Figure 10. Receiver Operating Characteristics; Example B

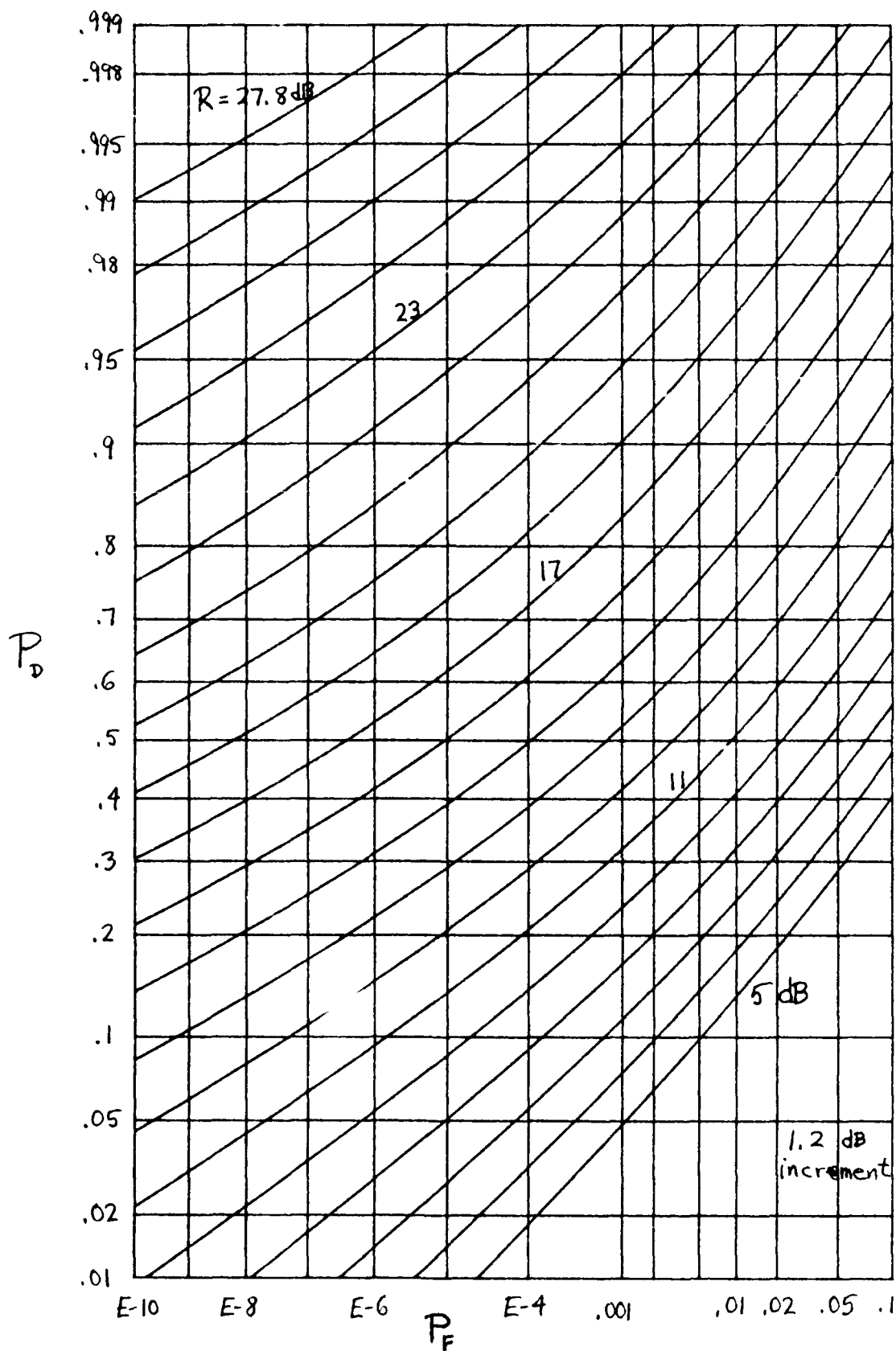


Figure 11. Receiver Operating Characteristics; Example C

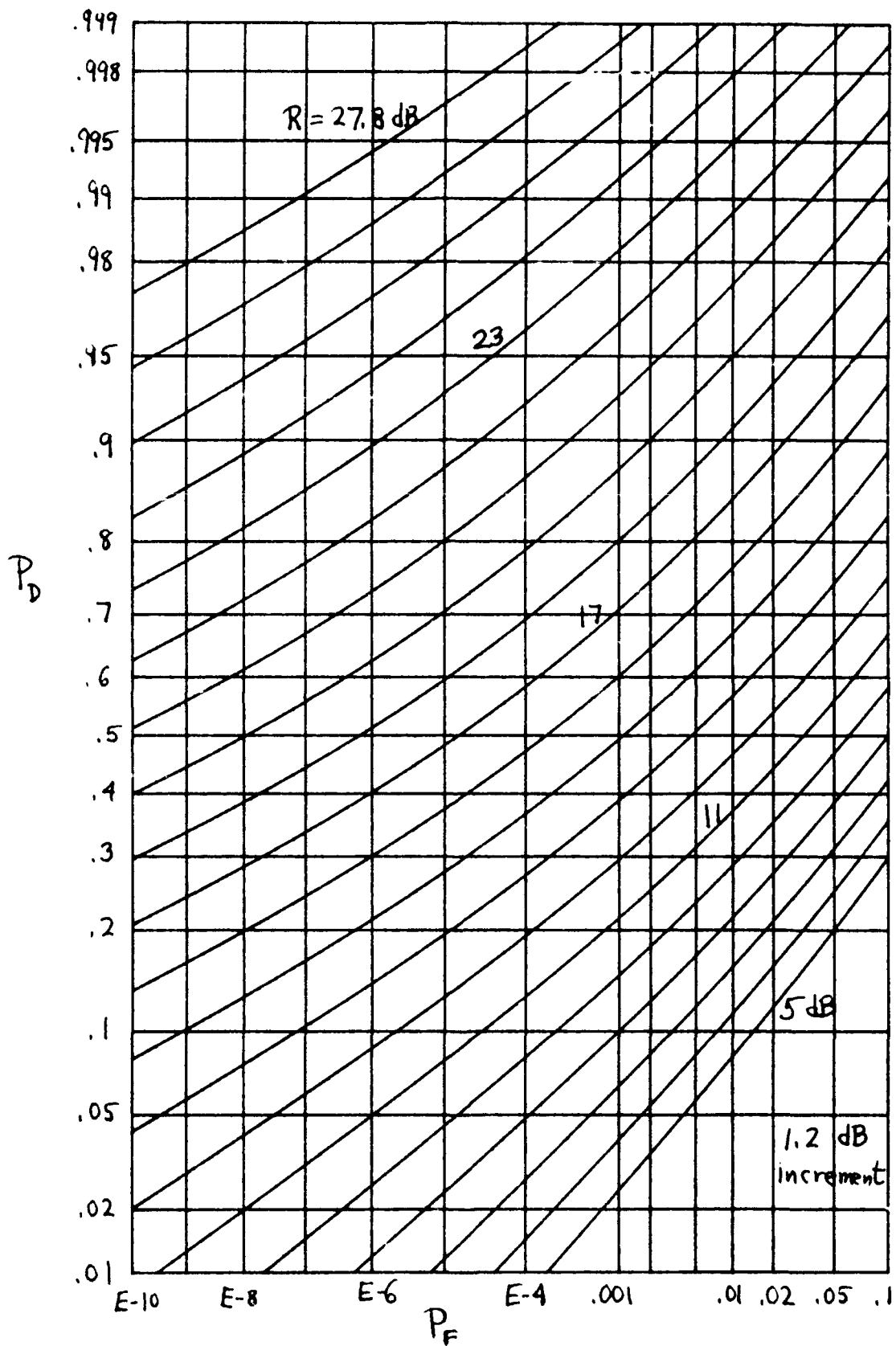


Figure 12. Receiver Operating Characteristics; Example D

## SUMMARY

Programs have been written which enable exact analysis of performance of the mismatched bandpass energy detector of figure 1, both for the white noise input case as well as the colored noise input case. These results allow for arbitrary sampling time increment  $\Delta$  and for arbitrary input signal spectra, input noise spectra, and filter transfer functions. By these means, exact quantitative evaluations and degradations can be determined for various combinations of uncertainty regarding the input signal time location and duration as well as its center frequency, bandwidth, and spectrum. Included in the analysis and programs are the buildup and decay (or any portion thereof) of the nonstationary output signal from the filter, when excited by a burst-like input signal, regardless of the lengths or locations of the excitation and observation intervals. Both programs have been compared with simulation results and confirmed down to as low a probability level as possible consistent with the number of trials used.

APPENDIX A. NUMERICAL EVALUATION OF NOISE COVARIANCE  $R_n$ 

If the noise covariance  $R_n(\tau)$  is not available in analytic form, but the noise spectrum  $G_n(f)$  is specified, the following numerical procedure can be employed to evaluate the required samples  $R_n(k\Delta)$ . We begin with

$$R_n(\tau) = \int df \exp(i2\pi f\tau) G_n(f) =$$

$$\approx \Delta_f \sum_m \exp(i2\pi m\Delta_f\tau) G_n(m\Delta_f) \equiv \tilde{R}_n(\tau) , \quad (A-1)$$

where the trapezoidal rule with frequency increment  $\Delta_f$  was used. But  $\tilde{R}_n(\tau)$  in (A-1) can be developed as

$$\tilde{R}_n(\tau) = \int df \exp(i2\pi f\tau) G_n(f) \Delta_f \delta_{\Delta_f}(f) =$$

$$= R_n(\tau) \otimes \delta_{1/\Delta_f}(\tau) = \sum_m R_n\left(\tau - \frac{m}{\Delta_f}\right) . \quad (A-2)$$

Now suppose that the noise covariance

$$R_n(\tau) \approx 0 \quad \text{for } |\tau| > \tau_0 . \quad (A-3)$$

Then, in order to avoid significant aliasing in (A-2), we must take  $\Delta_f$  small enough that

$$\frac{1}{2\Delta_f} > \tau_0 , \quad (A-4)$$

in which case (A-2) yields

$$\tilde{R}_n(\tau) \approx R_n(\tau) \quad \text{for } |\tau| < \tau_0 < \frac{1}{2\Delta_f} . \quad (A-5)$$



The required samples of  $\tilde{R}_n(\tau)$  follow from (A-1) according to

$$\tilde{R}_n(k\Delta) = \Delta_f \sum_m \exp(i2\pi m \Delta_f k \Delta) G_n(m \Delta_f) . \quad (A-6)$$

Now take frequency increment

$$\Delta_f = \frac{1}{N\Delta} , \quad (A-7)$$

where, according to (A-4), integer N must be taken large enough that  $N\Delta/2 > \tau_0$ , that is,

$$N > \frac{2\tau_0}{\Delta} . \quad (A-8)$$

Then, from (A-5) - (A-7),

$$R_n(k\Delta) \approx \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_m \exp(i2\pi mk/N) G_n\left(\frac{m}{N\Delta}\right) \text{ for } |k| < \frac{N}{2} . \quad (A-9)$$

Alternatively, this can be expressed as

$$R_n(k\Delta) \approx \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_{m=0}^{N-1} \exp(i2\pi mk/N) \tilde{G}_n\left(\frac{m}{N\Delta}\right) \text{ for } |k| < \frac{N}{2} , \quad (A-10)$$

where collapsed (prealiased) noise spectrum and sequence

$$\tilde{G}_n(f) \equiv \sum_j G_n\left(f - \frac{j}{\Delta}\right) , \quad \tilde{G}_n\left(\frac{m}{N\Delta}\right) = \sum_j G_n\left(\frac{m}{N\Delta} - \frac{j}{\Delta}\right) \text{ for } 0 \leq m \leq N-1 . \quad (A-11)$$

The procedure in (A-10) can be efficiently realized as an N-point FFT of N nonzero numbers. The major conditions that must be met are (A-8) in conjunction with (A-3).

APPENDIX B. NUMERICAL EVALUATION OF FILTER CORRELATION  $\phi_h$ 

Suppose that we want digital filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta) h(m\Delta - j\Delta) , \quad (B-1)$$

defined in (10), but all that we have is the filter transfer function  $H(f)$ , where

$$h(\tau) = \int df \exp(i2\pi f\tau) H(f) \quad \text{for all } \tau . \quad (B-2)$$

Impulse response  $h(\tau)$  is presumed real. The direct relationship between  $\phi_h(j\Delta)$  and  $H(f)$  is the subject of this appendix.

First, define periodic function

$$\tilde{H}(f) \equiv \sum_m H\left(f - \frac{m}{\Delta}\right) \quad \text{for all } f . \quad (B-3)$$

Then, there follows

$$\begin{aligned} \tilde{H}(f) &= H(f) \oplus \delta_{1/\Delta}(f) = \\ &= \int d\tau \exp(-i2\pi f\tau) h(\tau) \Delta \delta_{\Delta}(\tau) = \sum_m \exp(-i2\pi f\Delta m) \Delta h(m\Delta) , \end{aligned} \quad (B-4)$$

where we used the fact that convolution in the frequency domain is equivalent to Fourier transformation of a product in the time domain. That is,  $\tilde{H}(f)$  is the Fourier transform of the samples of impulse response  $h(\tau)$  taken at time increment  $\Delta$ . However, care must be taken at points of discontinuity of  $h(\tau)$ ; for example, if  $h(\tau)$  is discontinuous at  $k\Delta$ , the contribution to the right-hand

side of (B-4), at  $m = k$ , is  $\frac{1}{2}[h(k\Delta+) + h(k\Delta-)]$ . This point and an example are discussed more fully in appendix C.

Now, there follows from (B-1) and (B-2),

$$\begin{aligned}
 \phi_h(j\Delta) &= \Delta^2 \sum_m \int df \exp(i2\pi f\Delta m) H(f) \int du \exp[-i2\pi u\Delta(m-j)] H^*(u) = \\
 &= \Delta \iint df du H(f) H^*(u) \exp(i2\pi u\Delta j) \Delta \sum_m \exp[i2\pi(f - u)\Delta m] = \\
 &= \Delta \iint df du H(f) H^*(u) \exp(i2\pi u\Delta j) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) = \\
 &= \Delta \sum_m \int df H(f) H^*\left(f - \frac{m}{\Delta}\right) \exp\left[i2\pi\left(f - \frac{m}{\Delta}\right)\Delta j\right] = \\
 &= \Delta \int df \exp(i2\pi f\Delta j) H(f) \sum_m H^*\left(f - \frac{m}{\Delta}\right) = \\
 &= \Delta \int df \exp(i2\pi f\Delta j) H(f) \tilde{H}^*(f) , \tag{B-5}
 \end{aligned}$$

where we used (B-3).

Now, let  $I_n$  denote the following interval of length  $1/\Delta$  on the  $f$  axis:

$$I_n \equiv \left(\frac{n-\frac{1}{2}}{\Delta}, \frac{n+\frac{1}{2}}{\Delta}\right) . \tag{B-6}$$

Then, the filter correlation  $\phi_h$  follows from (B-5) according to

$$\phi_h(j\Delta) = \Delta \sum_n \int_{I_n} df \exp(i2\pi f\Delta j) H(f) \tilde{H}^*(f) =$$

$$\begin{aligned}
&= \Delta \sum_n \int_{I_0} du \exp\left[i2\pi\left(u + \frac{n}{\Delta}\right)\Delta j\right] H\left(u + \frac{n}{\Delta}\right) \tilde{H}^*(u) = \\
&= \Delta \int_{I_0} du \exp(i2\pi u \Delta j) \tilde{H}^*(u) \sum_n H\left(u + \frac{n}{\Delta}\right) = \\
&= \Delta \int_{-.5/\Delta}^{.5/\Delta} du \exp(i2\pi u \Delta j) |\tilde{H}(u)|^2, \tag{B-7}
\end{aligned}$$

where we let  $u = f - n/\Delta$  and used (B-3) and the periodicity of  $\tilde{H}(f)$ . This result indicates that we must first alias the given transfer function  $H(f)$  according to (B-3) and then Fourier transform its magnitude-square over an interval of length  $1/\Delta$ .

An alternative approach that leads to the same result (B-7) is to use (B-2) and (B-3) in the form

$$h(m\Delta) = \int_{I_0} df \exp(i2\pi f \Delta m) H(f) = \int_{I_0} df \exp(i2\pi f \Delta m) \tilde{H}(f). \tag{B-8}$$

Use of this latter result in (B-1) leads to filter correlation

$$\begin{aligned}
\phi_h(j\Delta) &= \Delta^2 \sum_m \int_{I_0} df \exp(i2\pi f \Delta m) \tilde{H}(f) \int_{I_0} du \exp[-i2\pi u \Delta(m-j)] \tilde{H}^*(u) = \\
&= \Delta \iint_{I_0} df du \exp(i2\pi u \Delta j) \tilde{H}(f) \tilde{H}^*(u) \Delta \sum_m \exp[i2\pi(f - u)\Delta m] = \\
&= \Delta \iint_{I_0} df du \exp(i2\pi u \Delta j) \tilde{H}(f) \tilde{H}^*(u) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =
\end{aligned}$$

$$= \Delta \int_{I_0} df \exp(i2\pi f \Delta j) |\tilde{H}(f)|^2, \quad (B-9)$$

since only one impulse, at  $u = f$ , lies in interval  $I_0$ . Result (B-9) is identical to (B-7). Notice that we do not get

$$\Delta \int df \exp(i2\pi f \Delta j) |H(f)|^2 = \Delta \int_{I_0} df \exp(i2\pi f \Delta j) \sum_n \left| H\left(f + \frac{n}{\Delta}\right) \right|^2. \quad (B-10)$$

As for the actual numerical evaluation of (B-9), suppose we sample the integral on  $f$  with increment  $\Delta_f = 1/(K\Delta)$  and use the trapezoidal approximation; then there follows

$$\phi_h(j\Delta) \approx \frac{1}{K} \sum_{k=-K/2}^{K/2} w_k \exp\left(i2\pi \frac{k}{K} j\right) \left| \tilde{H}\left(\frac{k}{K\Delta}\right) \right|^2, \quad (B-11)$$

where weights  $\{w_k\}$  are given by

$$w_k = \begin{cases} \frac{1}{2} & \text{for } |k| = K/2 \\ 1 & \text{for } |k| < K/2 \end{cases}. \quad (B-12)$$

By using the periodicity  $K$  of the  $\exp$  and  $\tilde{H}$  terms in (B-11), that sum may be written exactly in the standard FFT form

$$\phi_h(j\Delta) \approx \frac{1}{K} \sum_{k=0}^{K-1} \exp(i2\pi j k / K) \left| \tilde{H}\left(\frac{k}{K\Delta}\right) \right|^2 \quad \text{for } |j| < \frac{K}{2}. \quad (B-13)$$

This is a good approximation, provided that the FFT size  $K$  satisfies  $K > 2\tau_h/\Delta$ , where  $\tau_h$  is the effective duration of filter correlation  $\phi_h(\tau)$ .

# APPENDIX C. DISPLACED SAMPLING AND FOURIER TRANSFORM OF DISCONTINUOUS FUNCTION

## GENERAL RELATIONS

Let  $a(t)$  and  $A(f)$  be a Fourier transform pair:

$$A(f) = \int dt \exp(-i2\pi ft) a(t) . \quad (C-1)$$

Also, let  $b(t)$  and  $B(f)$  be a Fourier transform pair. Then, Parseval's theorem states that

$$\int dt a(t) b^*(t) = \int df A(f) B^*(f) . \quad (C-2)$$

We now apply this relation to the case where

$$b(t) = \Delta \delta_{\Delta}(t - \alpha) , \quad B(f) = \delta_{1/\Delta}(f) \exp(-i2\pi f\alpha) . \quad (C-3)$$

There follows

$$L \equiv \Delta \sum_n a(n\Delta + \alpha) = \int dt a(t) \Delta \delta_{\Delta}(t - \alpha) = \quad (C-4)$$

$$= \int df A(f) \delta_{1/\Delta}(f) \exp(i2\pi f\alpha) = \sum_n A\left(\frac{n}{\Delta}\right) \exp(i2\pi n\alpha/\Delta) \equiv R . \quad (C-5)$$

If the samples required in the sums in (C-4) and (C-5) encounter a discontinuity of  $a(t)$  or  $A(f)$ , the value used in (C-4) or (C-5) must be taken as the average value approached from both sides of the discontinuity. Also, the sums in (C-4) and (C-5) may have to be interpreted as principal value, if necessary for convergence.

## EXAMPLE

$$a(t) = \begin{cases} \exp[-(b+ic)t] & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad b > 0. \quad (C-6)$$

$$A(f) = \frac{1}{b + i(c + 2\pi f)} \quad \text{for all } f. \quad (C-7)$$

Function  $a(t)$  is discontinuous at  $t = 0$ , while  $A(f)$  is continuous for all  $f$ .

Let  $0 < \alpha < \Delta$ ; then the sum in (C-4) is

$$L = \Delta \sum_{n=0}^{\infty} \exp[-(b+ic)(n\Delta+\alpha)] = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta. \quad (C-8)$$

At the same time, the sum in (C-5) is

$$R = \sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} \quad \text{for all } \alpha. \quad (C-9)$$

When  $\alpha \neq 0$  (or an integer multiple of  $\Delta$ ), the phase factor in the numerator of (C-9) yields a convergent sum for the real and imaginary parts, without the need for a principal value interpretation. It has also been verified numerically that (C-8) and (C-9) are equal, as predicted by (C-4) and (C-5); that is,

$$\sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta. \quad (C-10)$$

On the other hand, if  $\alpha = 0$ , then the sum in (C-4) is, accounting for the discontinuity of  $a(t)$ ,

$$L = \frac{\Delta}{2} + \Delta \sum_{n=1}^{\infty} \exp[-(b+ic)n\Delta] = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \quad (C-11)$$

Also, then, the sum in (C-5) is

$$R = \sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \sum_n \frac{b - i(c + 2\pi n/\Delta)}{b^2 + (c + 2\pi n/\Delta)^2} , \quad (C-12)$$

the imaginary part of which must be interpreted as a principal value sum. Again, it has been numerically verified (next page) that (C-11) and (C-12) are equal; that is,

$$\sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \quad (C-13)$$

Notice that the limit of (C-10) as  $\alpha \rightarrow 0+$  does not yield (C-13).

If we apply result (C-11) to  $\tilde{H}(f)$  in the last entry in (B-4), we have, for example (304),

$$\tilde{H}(f) = \frac{a}{2} \frac{1 + \exp(-a-i2\pi f\Delta)}{1 - \exp(-a-i2\pi f\Delta)} . \quad (C-14)$$

By contrast, the limit of (C-10) as  $\alpha \rightarrow 0+$  yields

$$\frac{a}{1 - \exp(-a-i2\pi f\Delta)} . \quad (C-15)$$



## BASIC PROGRAM FOR NUMERICAL VERIFICATION

```

10   Bs=.57                      | b
20   Cs=.71                      | c
30   De=.93                      | delta
40   Al=.1                      | alpha
50   PRINT Bs,Cs,De,Al
60   CALL Exp(-Al*Bs,-Al*Cs,Nr,Ni)
70   CALL Exp(-De*Bs,-De*Cs,Er,Ei)
80   CALL Div(De*Nr,De*Ni,1-Er,-Ei,Lr,Li)
90   IF Al=0. THEN Lr=Lr-.5*De
100  PRINT Lr,Li                | LEFT-HAND SIDE L
110  A=2*PI/De
120  B=A*Al
130  B2=B*B
140  Rr=Ri=0.
150  DOUBLE Ns                  | INTEGER
160  FOR Ns=-1E5 TO 1E5        | PRINCIPAL VALUE SUM
170  T=B*Ns
180  C=COS(T)
190  S=SIN(T)
200  T=Cs+A*Ns
210  D=D2+T*T
220  Dr=(C*Bs+S*T)/D
230  Di=(S*Bs-C*T)/D
240  Rr=Rr+Dr
250  Ri=Ri+Di
260  NEXT Ns
270  PRINT Rr,Ri                | RIGHT-HAND SIDE R
280  END
290  !
300  SUB Div(X1,Y1,X2,Y2,U,V)  | Z1/Z2
310  T=X2*X2+Y2*Y2
320  U=(X1*X2+Y1*Y2)/T
330  V=(Y1*X2-X1*Y2)/T
340  SUBEND
350  !
360  SUB Exp(X,Y,U,V)          | EXP(Z)
370  E=EXP(X)
380  U=E*COS(Y)
390  V=E*SIN(Y)
400  SUBEND

```

.57	.71	.93	0
.729357897734	-.80564044434		
.729357647983	-.805640755434		
.57	.71	.93	.001
1.19310532814	-.806028668009		
1.19353711305	-.806028668087		
.57	.71	.93	.1
1.07135515262	-.839119622089		
1.07135537888	-.839119622093		

## APPENDIX D. PROGRAMS FOR WHITE INPUT NOISE

Three BASIC programs are listed in this appendix. All the signal, noise, and filter parameters are restricted to be real here. The first program computes the filter output signal and noise covariance matrices for a white noise input, the sums of these matrices scaled by the various signal-to-noise ratios and then weighted, and the eigenvalues which are then stored. The particular subroutine listed here, SUB Svd, actually computes the eigenvectors in addition; a faster alternative would be to replace this subroutine by one which calculates only the eigenvalues. Twenty different nonzero values of the input signal-to-noise ratio are allowed in the program and must be chosen and input by the user.

These eigenvalues serve as the input to the second program which computes the cumulative and exceedance distribution functions according to the method given in [2] and then plots the receiver operating characteristics by elimination of the threshold variable according to the method described in (73) - (80) in the main text.

The third program is the simulation program used to check the two programs above. It is written for single-pole processes and allows for a colored input noise process but could be easily modified to handle more general processes. A run of  $1E6$  trials took 7 hours on the Hewlett-Packard 9000 computer.

```

10  ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES FOR WHITE
20  ! NOISE INPUT; STORE EIGENVALUES IN "EIG" IN LINE 1720
30  Ka=-10          ! INPUT SIGNAL START
40  Kb=5            ! INPUT SIGNAL END; Kb >= Ka
50  Kc=0            ! ACCUMULATOR START
60  Kd=5            ! ACCUMULATOR END; Kd >= Kc,Ka
70  Delta=.2        ! TIME SAMPLING INCREMENT (SECONDS)
80  J=3             ! NUMBER OF SIGNAL COMPONENTS
90  DATA 11.,-48.,75. ! SIGNAL SCALINGS (WE SET SUM = 1)
100 DATA 1.,.125,.066667 ! SIGNAL TIME CONSTANTS (SECONDS)
110 N=4             ! NUMBER OF FILTER COMPONENTS
120 DATA 1.,2.,3.,4. ! FILTER SCALINGS (ARBITRARY)
130 DATA .3,.5,.7,.9 ! FILTER TIME CONSTANTS (SECONDS)
140 Nr=20           ! NUMBER OF SIGNAL-TO-NOISE RATIOS
150 DATA 0,1,2,3,4,5,6,7,8,9,10 ! INPUT SNRS IN DB
160 DATA 11,12,13,14,15,16,17,18,19
170 IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 200
180 PRINT "PROBLEM WITH PARAMETERS"
190 STOP
200 DIM Alpha(10),A(10),Psi(10),C(10),R(20),W(100)
210 DIM Pc(10),Ec(10),Pcv(10),Cn(200),Ea(10),E(0,1000)
220 DIM Gca(100),Gcda(100),Gcb(100),Gbda(100),Cs(5000)
230 DIM Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
240 DOUBLE Ka,Kb,Kc,Kd,J,N,Nr,Ns,Ks,Kdc,Ko ! INTEGERS
250 DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
260 REDIM Alpha(1:J),A(1:J)
270 READ Alpha(*),A(*)
280 MAT Alpha=Alpha/(SUM(Alpha))
290 MAT A=(Delta)/A
300 REDIM Psi(1:N),C(1:N)
310 READ Psi(*),C(*)
320 MAT C=(Delta)/C
330 REDIM R(1:Nr)
340 READ R(*)
350 REDIM W(Kc:Kd)
360 CALL Weights(Kc,Kd,W(*))
370 MAT W=SQR(W)
380 REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
390 FOR Ns=1 TO N
400 C=C(Ns)
410 Pc(Ns)=Psi(Ns)*C
420 Ec(Ns)=EXP(-C)
430 NEXT Ns
440 REDIM Pcv(1:N)
450 FOR Ns=1 TO N
460 E=Ec(Ns)
470 S=0.
480 FOR Ks=1 TO N
490 S=C+Pc(Ks)/(1.-E*Ec(Ks))
500 NEXT Ks
510 Pcv(Ns)=Pc(Ns)*S
520 NEXT Ns
530 Kdc=Kd-Kc
540 REDIM Cn(0:Kdc)
550 MAT Cn=(0.)
560 FOR Ns=1 TO N
570 Pcv=Pcv(Ns)
580 E=Ec(Ns)

```

```

590 P=1.
600 Cn(0)=Cn(0)+Pcv
610 FOR Ks=1 TO Kdc
620 P=P*E
630 Cn(Ks)=Cn(Ks)+Pcv*P
640 NEXT Ks
650 NEXT Ns
660 FOR Js=1 TO J          ! FILTER OUTPUT SIGNAL COVARIANCE
670 A(Js)=EXP(A(Js))
680 NEXT Js
690 Ko=MAX(Ka,Kc)
700 REDIM Ea(1:N),E(1:N,Ko:Kd)
710 FOR Ns=1 TO N
720 E=Ec(Ns)
730 Ea(Ns)=1./E
740 E(Ns,Ko)=EXP(-C(Ns)*Ko)
750 FOR Ks=Ko+1 TO Kd
760 E(Ns,Ks)=E(Ns,Ks-1)*E
770 NEXT Ks
780 NEXT Ns
790 L1=MIN(Ko,Kb)+1
800 L2=MIN(Kd,Kb)+1
810 REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
820 L=Kd-Ko+1
830 REDIM Cs(1:L*(L+1)/2)
840 L11=L1+1
850 FOR Js=1 TO J
860 A1=Alpha(Js)
870 A=A(Js)
880 A1=A*A-1.
890 FOR Ns=1 TO N
900 C=Ea(Ns)
910 Ca=C*A
920 Cda=C/A
930 Ca1=1.-Ca
940 Ac=A-C
950 Ae=A1*Pc(Ns)
960 Gca=Ca^Ka/Ca1
970 Gca(L1)=Ca^L1/Ca1
980 Gcda(L1)=Cda^L1/(1.-Cda)
990 FOR Ks=L11 TO L2
1000 Gca(Ks)=Ca*Gca(Ks-1)
1010 Gcda(Ks)=Cda*Gcda(Ks-1)
1020 NEXT Ks
1030 FOR Qs=1 TO N
1040 B=Ea(Qs)
1050 Aee=Ae*Pc(Qs)
1060 Ba=B*A
1070 Bda=B/A
1080 Cb=C*B
1090 Ba1=1.-Ba
1100 Cb1=1.-Cb
1110 Gba=Ba^Ka/Ba1
1120 Ab=A-B
1130 F=B*A1/(Ab*Ba1)
1140 S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1150 Gcb(L1)=Cb^L1/Cb1
1160 Gbda(L1)=Bda^L1/(1.-Bda)

```

```

1170 FOR Ks=L11 TO L2
1180 Gcb(Ks)=Cb*Gcb(Ks-1)
1190 Gbda(Ks)=Bda*Gbda(Ks-1)
1200 NEXT Ks
1210 Ki=0
1220 FOR Ks1=Kc TO Kd
1230 K1=MIN(Ks1,Kb)+1
1240 S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1250 S3=Gca(K1)-Gca
1260 E=Aee*E(Ns,Ks1)
1270 FOR Ks2=Ks1 TO Kd
1280 Ki=Ki+1
1290 K2=MIN(Ks2,Kb)+1
1300 S=S2+Gbda(K2)*S3 ! SUM S
1310 Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*3
1320 NEXT Ks2
1330 NEXT Ks1
1340 NEXT Qs
1350 NEXT Ns
1360 NEXT Js
1370 L=Kd-Kc+1 ! TOTAL WEIGHTED COVARIANCE MATRIX
1380 REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1390 L11=Kc-1
1400 R=0.
1410 FOR I=0 TO Nr
1420 Ki=0
1430 IF I=0 THEN 1450
1440 R=10.^(R(I)*.1) ! INPUT POWER SIGNAL-TO-NOISE RATIO
1450 FOR Ks1=Kc TO Kd
1460 W=W(Ks1) ! WEIGHTS (w(k))
1470 FOR Ks2=Ks1 TO Kd
1480 IF Ks1=Ko THEN 1510
1490 Cs=0.
1500 GOTO 1530
1510 Ki=Ki+1
1520 Cs=Cc(Ki)
1530 Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
1540 L1=Ks1-L11
1550 L2=Ks2-L11
1560 Cw(L1,L2)=Cw(L2,L1)=Pr
1570 NEXT Ks2
1580 NEXT Ks1
1590 CALL Svd(L,L,Cw(*),Ca(*),D(*)) ! EIGENVALUES
1600 MAT SORT D(*) DES
1610 IF D(L)>0. THEN 1640
1620 PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
1630 PAUSE
1640 PRINT "I =";I;" CONDITION NUMBER =";D(1)/D(L)
1650 PRINT D(*)
1660 PRINT
1670 FOR Ks=1 TO L
1680 Eig(I,Ks)=D(Ks) ! STORE EIGENVALUES
1690 NEXT Ks
1700 NEXT I
1710 MASS STORAGE IS ";CS80,7"
1720 ASSIGN #1 TO "EIG"
1730 PRINT #1;Nr,Kc,Kd,Eig(*)
1740 ASSIGN #1 TO *
1750 END
1760 !

```

```

1770 SUB Weights(DOUBLE Kc,Kd,REAL W(*))
1780 DOUBLE Ks ! INTEGER
1790 F=1. ! DECAY FACTOR (DIMENSIONLESS)
1800 W(Kd)=1.
1810 FOR Ks=Kd-1 TO Kc STEP -1
1820 W(Ks)=W(Ks+1)*F ! EXPONENTIAL WEIGHTS
1830 NEXT Ks
1840 SUBEND
1850 !
1860 SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
1870 ALLOCATE Rv1(1:N) ! NUMERICAL RECIPES, PAGES 60-64
1880 IF M>N THEN 1910 ! A(*) IS OVER-WRITTEN
1890 PRINT "M<N IS DISALLOWED"
1900 PAUSE
1910 DOUBLE I,J,K,L,Its,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
1920 G=Scale=Anorm=0.
1930 FOR I=1 TO N
1940 L=I+1
1950 Rv1(I)=Scale*G
1960 G=S=Scale=0.
1970 IF I>M THEN 2250
1980 FOR K=I TO M
1990 Scale=Scale+ABS(A(K,I))
2000 NEXT K
2010 IF Scale=0. THEN 2250
2020 FOR K=I TO M
2030 Ra=A(K,I)=A(K,I)/Scale
2040 S=S+Ra*Ra
2050 NEXT K
2060 F=A(I,I)
2070 G=-SQR(S)
2080 IF F<0. THEN G=-G
2090 H=F*G-S
2100 A(I,I)=F-G
2110 IF I=N THEN 2220
2120 FOR J=L TO N
2130 S=0.
2140 FOR K=I TO M
2150 S=S+A(K,I)*A(K,J)
2160 NEXT K
2170 F=S/H
2180 FOR K=I TO M
2190 A(K,J)=A(K,J)+F*A(K,I)
2200 NEXT K
2210 NEXT J
2220 FOR K=I TO M
2230 A(K,I)=A(K,I)*Scale
2240 NEXT K
2250 W(I)=Scale*G
2260 G=S=Scale=0.
2270 IF (I>M) OR (I=N) THEN 2570
2280 FOR K=L TO N
2290 Scale=Scale+ABS(A(I,K))
2300 NEXT K
2310 IF Scale=0. THEN 2570

```

```

2320   FOR K=L TO N
2330   Ra=A(I,K)=A(I,K)/Scale
2340   S=S+Ra*Ra
2350   NEXT K
2360   F=A(I,L)
2370   G=-SQR(S)
2380   IF F<0. THEN G=-G
2390   H=F*G-S
2400   A(I,L)=F-G
2410   FOR K=L TO N
2420   Rv1(K)=A(I,K)/H
2430   NEXT K
2440   IF I=M THEN 2540
2450   FOR J=L TO M
2460   S=0.
2470   FOR K=L TO N
2480   S=S+A(J,K)*A(I,K)
2490   NEXT K
2500   FOR K=L TO N
2510   A(J,K)=A(J,K)+S*Rv1(K)
2520   NEXT K
2530   NEXT J
2540   FOR K=L TO N
2550   A(I,K)=A(I,K)*Scale
2560   NEXT K
2570   Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
2580   NEXT I
2590   FOR I=N TO 1 STEP -1
2600   IF I>=N THEN 2770
2610   IF G=0. THEN 2740
2620   FOR J=L TO N
2630   V(J,I)=A(I,J)/A(I,L)/G
2640   NEXT J
2650   FOR J=L TO N
2660   S=0.
2670   FOR K=L TO N
2680   S=S+A(I,K)*V(K,J)
2690   NEXT K
2700   FOR K=L TO N
2710   V(K,J)=V(K,J)+S*V(K,I)
2720   NEXT K
2730   NEXT J
2740   FOR J=L TO N
2750   V(I,J)=V(J,I)=0.
2760   NEXT J
2770   V(I,I)=1.
2780   G=Rv1(I)
2790   L=I
2800   NEXT I
2810   FOR I=N TO 1 STEP -1
2820   L=I+1
2830   G=W(I)
2840   IF I>=N THEN 2880
2850   FOR J=L TO N
2860   A(I,J)=0.
2870   NEXT J

```

```

2880 IF G=0. THEN 3050
2890 G=1./G
2900 IF I=N THEN 3010
2910 FOR J=L TO N
2920 S=0.
2930 FOR K=L TO M
2940 S=S+A(K,I)*A(K,J)
2950 NEXT K
2960 F=S/A(I,I)*G
2970 FOR K=I TO M
2980 A(K,J)=A(K,J)+F*A(K,I)
2990 NEXT K
3000 NEXT J
3010 FOR J=I TO M
3020 A(J,I)=A(J,I)*G
3030 NEXT J
3040 GOTO 3080
3050 FOR J=I TO M
3060 A(J,I)=0.
3070 NEXT J
3080 A(I,I)=A(I,I)+1.
3090 NEXT I
3100 FOR K=N TO 1 STEP -1
3110 FOR Its=1 TO 30
3120 FOR L=K TO 1 STEP -1
3130 Nm=L-1
3140 IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3360
3150 IF (ABS(W(Nm))+Anorm)=Anorm THEN 3170
3160 NEXT L
3170 C=0.
3180 S=1.
3190 FOR I=L TO K
3200 F=S*Rv1(I)
3210 Rv1(I)=C*Rv1(I)
3220 IF (ABS(F)+Anorm)=Anorm THEN 3360
3230 G=W(I)
3240 H=SQR(F*F+G*G)
3250 W(I)=H
3260 H=1./H
3270 C=G*H
3280 S=-F*H
3290 FOR J=1 TO M
3300 Y=A(J,Nm)
3310 Z=A(J,I)
3320 A(J,Nm)=Y*C+Z*S
3330 A(J,I)=-Y*S+Z*C
3340 NEXT J
3350 NEXT I
3360 Z=W(K)
3370 IF L<>K THEN 3440
3380 IF Z>=0. THEN 3430
3390 W(K)=-Z
3400 FOR J=1 TO N
3410 V(J,K)=-V(J,K)
3420 NEXT J
3430 GOTO 3970

```



```

3440 IF Its<30 THEN 3470
3450 PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3460 PAUSE
3470 X=W(L)
3480 Nm=K-1
3490 Y=W(Nm)
3500 G=Rv1(Nm)
3510 H=Rv1(K)
3520 F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3530 G=SQR(F*F+1.)
3540 Aa=ABS(G)
3550 IF F<0, THEN Aa=-Aa
3560 F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3570 C=S=1.
3580 FOR J=L TO Nm
3590 I=J+1
3600 G=Rv1(I)
3610 Y=W(I)
3620 H=S*G
3630 G=C*G
3640 Z=SQR(F*F+H*H)
3650 Rv1(J)=Z
3660 C=F/Z
3670 S=H/Z
3680 F=X*C+G*S
3690 G=-X*S+G*C
3700 H=Y*S
3710 Y=Y*C
3720 FOR Jj=1 TO N
3730 X=V(Jj,J)
3740 Z=V(Jj,I)
3750 V(Jj,J)=X*C+Z*S
3760 V(Jj,I)=-X*S+Z*C
3770 NEXT Jj
3780 Z=SQR(F*F+H*H)
3790 W(J)=Z
3800 IF Z=0, THEN 3840
3810 Z=1./Z
3820 C=F*Z
3830 S=H*Z
3840 F=C*G+S*Y
3850 X=-S*G+C*Y
3860 FOR Jj=1 TO M
3870 Y=A(Jj,J)
3880 Z=A(Jj,I)
3890 A(Jj,J)=Y*C+Z*S
3900 A(Jj,I)=-Y*S+Z*C
3910 NEXT Jj
3920 NEXT J
3930 Rv1(L)=0.
3940 Rv1(K)=F
3950 W(K)=X
3960 NEXT Its
3970 NEXT K
3980 SUBEND

```

```

10 ! PLOT ROC FROM EIGENVALUES "EIG", LINES 120 & 1060
20 Delxi=.08 ! INITIAL INCREMENT ON CHAR. FUNCTION
30 M=128 ! INITIAL SIZE OF FFT
40 Bs=0. ! SHIFT b
50 Tol0=1.E-36 ! TOLERANCE FOR FALSE ALARM PROBABILITY
60 Tol1=1.E-32 ! TOLERANCE FOR DETECTION PROBABILITIES
70 DOUBLE N,Nr,Kc,Kd,Kdc,Ms,I,Ks,Ns,Im,Mm ! INTEGERS
80 DIM E(0:20,1:100),D(1:100),Cos(0:4096) ! 20 NONZERO SNRs
90 DIM X(0:16383),Y(0:16383),Pr(0:20,0:127)
100 PRINT "Delxi =";Delxi,"M =";M,"b =";Bs
110 MASS STORAGE IS ":CS80,7"
120 ASSIGN #1 TO "EIG"
130 READ #1;Nr,Kc,Kd ! NO. OF SNRs, START & END OF SUMMER
140 Kdc=Kd-Kc+1
150 REDIM E(0:Nr,1:Kdc),D(1:Kdc)
160 READ #1;E(*) ! EIGENVALUES
170 GINIT
180 PLOTTER IS "GRAPHICS"
190 GRAPHICS ON
200 REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
210 A=2.*PI/M
220 FOR Ms=0 TO M/4
230 Cos(Ms)=COS(A*Ms) ! INITIAL QUARTER-COSINE TABLE
240 NEXT Ms
250 Tol=Tol0
260 FOR I=0 TO Nr ! Nr NONZERO SIGNAL-TO-NOISE RATIOS
270 IF I>0 THEN Tol=Tol1
280 FOR Ks=1 TO Kdc
290 D(Ks)=E(I,Ks) ! COPY EIGENVALUES
300 NEXT Ks
310 Mux=SUM(D) ! MEAN OF RANDOM VARIABLE x
320 R=0. ! ARGUMENT OF SQUARE ROOT
330 P=1. ! POLARITY INDICATOR
340 Muy=Mux+Bs ! MEAN OF y = x + b
350 MAT X=(0.)
360 MAT Y=(0.)
370 Y(0)=.5*Delxi*Muy
380 Ns=0
390 Ns=Ns+1 ! LOOP ON xi
400 Xi=Delxi*Ns ! ARGUMENT xi OF CHAR. FH.
410 Txi=Xi*2. ! CALCULATION OF
420 Pr=1. ! CHARACTERISTIC
430 Pi=0. ! FUNCTION fy(xi)
440 FOR Ks=1 TO Kdc
450 Te=Txi*D(Ks)
460 Pe=Pr+Te*Pi
470 Pi=Pi-Te*Pr
480 Pr=Pe
490 NEXT Ks
500 CALL Sqr(Pr,Pi,Sr,Si)
510 De=Sr*Sr+Si*Si
520 Fyr=Sr/De
530 Fyi=-Si/De
540 Ro=R
550 R=ATN(Fyi/Fyr)
560 IF ABS(R-Ro)>1.6 THEN P=-P
570 IF P>0. THEN 600

```

```

580   Fyr=-Fyr
590   Fyi=-Fyi
600   Ms=Ns MODULO M           ! COLLAPSING
610   A=Fyr/Ns
620   B=Fyi/Ns
630   X(Ms)=X(Ms)+A
640   Y(Ms)=Y(Ms)+B
650   IF A*A+B*B<To1 THEN 670
660   GOTO 390
670   CALL Fft14(M,Cos(*),X(*),Y(*))
680   GCLEAR
690   WINDOW 0,M,-18,0
700   LINE TYPE 3
710   GRID M/8,1
720   LINE TYPE 1
730   FOR Ks=0 TO M-1
740   T=Y(Ks)/PI-Ks/M
750   Y(Ks)=Pr=.5+T           ! EXCEEDANCE PROBABILITY IN Y(*)
760   IF Pr>=1.E-16 THEN Y=LGT(Pr)!Y(Ks)=PROB(x>2 PI Ks/(M Delxi)-Bs)
770   IF Pr<=-1.E-16 THEN Y=-32.-LGT(-Pr)
780   IF ABS(Pr)<1.E-16 THEN Y=-16.
790   PLOT Ks,Y
800   NEXT Ks
810   PENUP
820   IF I=Nr THEN 940
830   PRINT I;
840   INPUT "SCALE FFT SIZE BY 1,2,4,8,...:",Im
850   Mm=MIN(M*Im,16384)
860   IF Mm=M THEN 940
870   Delxi=Delxi*M/Mm
880   M=Mm
890   REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
900   A=2.*PI/M
910   FOR Ms=0 TO M/4
920   Cos(Ms)=COS(A*Ms)       ! QUARTER-COSINE TABLE; <= 4096
930   NEXT Ms
940   FOR Ks=0 TO 127
950   Pr(I,Ks)=Y(Ks)         ! STORE FOR ROC PLOT
960   NEXT Ks
970   NEXT I
980   FOR Ks=0 TO 127
990   IF Pr(0,Ks)<.1 THEN 1010
1000  NEXT Ks
1010  Ms=Ks-1
1020  FOR Ks=Ms TO 127
1030  IF Pr(0,Ks)<1E-10 THEN 1050
1040  NEXT Ks
1050  Ms=Ks
1060  ASSIGN #1 TO "ROC"
1070  PRINT #1;Ms,Ns,Nr
1080  PRINT #1;Pr(*)
1090  ASSIGN #1 TO *
1100  DIM Pf(1:14),Pd(1:18)
1110  DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4
1120  DATA .001,.002,.005,.01,.02,.05,.1
1130  READ Pf(*)

```

```

1140 DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7
1150 DATA .8,.9,.95,.98,.99,.995,.998,.999
1160 READ Pd(*)
1170 FOR I=1 TO 14
1180 Pf(I)=FNInvphi(Pf(I))
1190 NEXT I
1200 FOR I=1 TO 18
1210 Pd(I)=FNInvphi(Pd(I))
1220 NEXT I
1230 GCLEAR
1240 X1=Pf(1)
1250 X2=Pf(14)
1260 Y1=Pd(1)
1270 Y2=Pd(18)
1280 WINDOW X1,X2,Y1,Y2
1290 LINE TYPE 3
1300 FOR I=1 TO 14
1310 MOVE Pf(I),Y1
1320 DRAW Pf(I),Y2
1330 NEXT I
1340 FOR I=1 TO 18
1350 MOVE X1,Pd(I)
1360 DRAW X2,Pd(I)
1370 NEXT I
1380 PENUP
1390 LINE TYPE 1
1400 FOR Ks=Ms TO Ns
1410 Pr(0,Ks)=FNInvphi(Pr(0,Ks)) ! FALSE ALARM PROBABILITY
1420 NEXT Ks
1430 FOR I=1 TO Nr
1440 FOR Ks=Ms TO Ns
1450 X=Pr(0,Ks)
1460 Pr=Pr(I,Ks)
1470 IF Pr>.9999 THEN 1510
1480 Pr(I,Ks)=Y=FNInvphi(Pr)
1490 PLOT X,Y
1500 IF Pr<.01 THEN 1520
1510 NEXT Ks
1520 PENUP
1530 NEXT I
1540 PAUSE
1550 END
1560 !
1570 DEF FNArg(X,Y) ! PRINCIPAL ARG(Z)
1580 IF X=0. THEN RETURN .5*PI*SGN(Y)
1590 A=ATN(Y/X)
1600 IF X>0. THEN RETURN A
1610 IF Y<0. THEN RETURN A-PI
1620 RETURN A+PI
1630 FNEND
1640 !
1650 SUB Sqr(X,Y,U,V) ! PRINCIPAL SQR(Z)
1660 F=SQR(SQR(X*X+Y*Y))
1670 T=.5*FNArg(X,Y)
1680 U=F*COS(T)
1690 V=F*SIN(T)
1700 SUBEND
1710 !

```

```

1720 DEF FNInvphi(X) ! AMS 55, 26.2.23
1730 IF X=.5 THEN RETURN 0.
1740 P=MIN(X,1.-X)
1750 P=MAX(P,1.E-20)
1760 T=-LOG(P)
1770 T=SQR(T+T)
1780 P=1.+T*(1.432788+T*(.189269+T*.001308))
1790 P=T-(2.515517+T*(.802853+T*.010328))/P
1800 IF X<.5 THEN P=-P
1810 RETURN P
1820 FNEND
1830 !
1840 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! .K=2^14=16384; 0 SUBS
1850 DOUBLE Log2n,N1,N2,N3,N4,J,K ! INTEGERS < 2^31 = 2,147,483,648
1860 DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1870 IF N=1 THEN SUBEXIT
1880 IF N>2 THEN 1960
1890 A=X(0)+X(1)
1900 X(1)=X(0)-X(1)
1910 X(0)=A
1920 A=Y(0)+Y(1)
1930 Y(1)=Y(0)-Y(1)
1940 Y(0)=A
1950 SUBEXIT
1960 A=LOG(N)/LOG(2.)
1970 Log2n=A
1980 IF ABS(A-Log2n)<1.E-8 THEN 2010
1990 PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
2000 PAUSE
2010 N1=N/4
2020 N2=N1+1
2030 N3=N2+1
2040 N4=N3+N1
2050 FOR I1=1 TO Log2n
2060 I2=2^(Log2n-I1)
2070 I3=2*I2
2080 I4=N/I3
2090 FOR I5=1 TO I2
2100 I6=(I5-1)*I4+1
2110 IF I6<=N2 THEN 2150
2120 A1=-Cos(N4-I6-1)
2130 A2=-Cos(I6-N1-1)
2140 GOTO 2170
2150 A1=Cos(I6-1)
2160 A2=-Cos(N3-I6-1)
2170 FOR I7=0 TO N-I3 STEP I3
2180 I8=I7+I5-1
2190 I9=I8+I2
2200 T1=X(I8)
2210 T2=X(I9)
2220 T3=Y(I8)
2230 T4=Y(I9)

```

```

2240      A3=T1-T2
2250      A4=T3-T4
2260      X(I8)=T1+T2
2270      Y(I8)=T3+T4
2280      X(I9)=A1*A3-A2*A4
2290      Y(I9)=A1*A4+A2*A3
2300      NEXT I7
2310      NEXT I5
2320      NEXT I1
2330      I1=Log2n+1
2340      FOR I2=1 TO 14
2350      L(I2-1)=1
2360      IF I2>Log2n THEN 2380
2370      L(I2-1)=2^(I1-I2)
2380      NEXT I2
2390      K=0
2400      FOR I1=1 TO L(13)
2410      FOR I2=I1 TO L(12) STEP L(13)
2420      FOR I3=I2 TO L(11) STEP L(12)
2430      FOR I4=I3 TO L(10) STEP L(11)
2440      FOR I5=I4 TO L(9) STEP L(10)
2450      FOR I6=I5 TO L(8) STEP L(9)
2460      FOR I7=I6 TO L(7) STEP L(8)
2470      FOR I8=I7 TO L(6) STEP L(7)
2480      FOR I9=I8 TO L(5) STEP L(6)
2490      FOR I10=I9 TO L(4) STEP L(5)
2500      FOR I11=I10 TO L(3) STEP L(4)
2510      FOR I12=I11 TO L(2) STEP L(3)
2520      FOR I13=I12 TO L(1) STEP L(2)
2530      FOR I14=I13 TO L(0) STEP L(1)
2540      J=I14-1
2550      IF K>J THEN 2620
2560      A=X(K)
2570      X(K)=X(J)
2580      X(J)=A
2590      A=Y(K)
2600      Y(K)=Y(J)
2610      Y(J)=A
2620      K=K+1
2630      NEXT I14
2640      NEXT I13
2650      NEXT I12
2660      NEXT I11
2670      NEXT I10
2680      NEXT I9
2690      NEXT I8
2700      NEXT I7
2710      NEXT I6
2720      NEXT I5
2730      NEXT I4
2740      NEXT I3
2750      NEXT I2
2760      NEXT I1
2770      SUBEND

```

```

10  ! SIMULATION FOR ONE-POLE PROCESSES
20  As=.85                ! INPUT SIGNAL PARAMETER
30  Bs=.77                ! INPUT NOISE PARAMETER; LINE 160
40  Cs=1.13              ! FILTER PARAMETER
50  R=.7                  ! INPUT POWER SNR
60  Ka=4
70  Kb=11
80  Kc=2
90  Kd=16
100 Bins=1000
110 Num=1E6               ! NUMBER OF TRIALS
120 Deltaz=.25            ! BIN SIZE FOR OUTPUT Z
130 DOUBLE Ka,Kb,Kc,Kd,Bins,Num,It,K,M
140 DIM E(100),S(100),Yn(100),Ys(100),T(1000)
150 Ea=EXP(-As)
160 Eb=EXP(-Bs)           ! WHITE NOISE: FOR Bs=inf, SET Eb=0
170 Ec=EXP(-Cs)
180 Fa=SQR(R*(1.-Ea*Ea))
190 Fb=SQR(1.-Eb*Eb)
200 REDIM E(0;Kd-Ka)
210 E(0)=Cs
220 FOR K=1 TO Kd-Ka
230 E(K)=E(K-1)*Ec
240 NEXT K
250 REDIM S(-30;Kd),Yn(Kc-1;Kd),Ys(Ka;Kd),T(1;Bins)
260 FOR It=1 TO Num
270 S(-30)=Nk=Yk=0.
280 FOR K=-29 TO Kc-1    ! ALLOW STEADY STATE
290 R1=2.*RND-1.         ! BOX-MULLER
300 R2=2.*RND-1.
310 R3=R1*R1+R2*R2
320 IF R3>1. THEN 290
330 R3=SQR(-2.*LOG(R3)/R3)
340 R1=R1*R3
350 R2=R2*R3
360 S(K)=Fa*R1+Ea*S(K-1) ! FILTER INPUT SIGNAL
370 Nk=Fb*R2+Eb*Nk       ! FILTER INPUT NOISE
380 Yk=Cs*Nk+Ec*Yk       ! FILTER OUTPUT NOISE
390 NEXT K
400 Yn(Kc-1)=Yk
410 FOR K=Kc TO Kd
420 R1=2.*RND-1.
430 R2=2.*RND-1.
440 R3=R1*R1+R2*R2
450 IF R3>1. THEN 420
460 R3=SQR(-2.*LOG(R3)/R3)
470 R1=R1*R3
480 R2=R2*R3
490 M=K-1

```

```

500  S(K)=Fa*R1+Ea*S(M)      !  FILTER INPUT  SIGNAL
510  Nk=Fc*R2+Eb*Nk         !  FILTER INPUT  NOISE
520  Yn(K)=Cs*Nk+Ec*Yn(M)   !  FILTER OUTPUT NOISE
530  NEXT K
540  FOR K=Ka TO Kd
550  S=0.
560  FOR M=Ka TO MIN(K,Kb)
570  S=S+E(K-M)*S(M)        !  FILTER OUTPUT SIGNAL
580  NEXT M
590  Ys(K)=S
600  NEXT K
610  Z=0.
620  FOR K=Kc TO Ka-1
630  T=Yn(K)
640  Z=Z+T*T
650  NEXT K
660  FOR K=Ka TO Kd
670  T=Ys(K)+Yn(K)
680  Z=Z+T*T
690  NEXT K
700  K=INT(Z/Deltaz)+1
710  K=MIN(K,Bins)
720  T(K)=T(K)+1.
730  NEXT K
740  FOR K=Bins-1 TO 1 STEP -1
750  T(K)=T(K)+T(K+1)
760  NEXT K
770  MAT T=T/(Num)
780  GINIT
790  PLOTTER IS "GRAPHICS"
800  GRAPHICS ON
810  WINDOW 0,Bins,-5,0
820  GRID Bins/4,1
830  FOR K=1 TO Bins
840  T=T(K)
850  IF T>0. THEN 870
860  GOTO 890
870  PLOT K,LGT(T)
880  NEXT K
890  PENUP
900  FOR K=Bins TO 1 STEP -1
910  T=1.-T(K)
920  IF T>0. THEN 940
930  GOTO 960
940  PLOT K,LGT(T)
950  NEXT K
960  PENUP
970  PAUSE
980  DUMP GRAPHICS
990  END

```



## APPENDIX E. PROGRAM FOR COLORED INPUT NOISE

One program is listed in this appendix. It computes the covariance matrices and eigenvalues as described in appendix D, but now for a colored noise input. The program for computing the receiver operating characteristics from the eigenvalues is identical to that listed above and therefore has not been repeated here.

```

10  ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES
20  ! STORE EIGENVALUES IN "EIG" IN LINE 2260
30  Ka=-10          ! INPUT SIGNAL START
40  Kb=5            ! INPUT SIGNAL END; Kb >= Ka
50  Kc=0           ! ACCUMULATOR START
60  Kd=5           ! ACCUMULATOR END; Kd >= Kc,Ka
70  Delta=.2       ! TIME SAMPLING INCREMENT (SECONDS)
80  J=3            ! NUMBER OF SIGNAL COMPONENTS
90  DATA 11.,-48.,75. ! SIGNAL SCALINGS (WE SET SUM = 1)
100 DATA 1.,.125,.066667 ! SIGNAL TIME CONSTANTS (SECONDS)
110 M=2            ! NUMBER OF NOISE COMPONENTS
120 DATA 39.,60.  ! NOISE SCALINGS (WE SET SUM = 1)
130 DATA .2,.4    ! NOISE TIME CONSTANTS (SECONDS)
140 N=4            ! NUMBER OF FILTER COMPONENTS
150 DATA 1.,2.,3.,4. ! FILTER SCALINGS (ARBITRARY)
160 DATA .3,.5,.7,.9 ! FILTER TIME CONSTANTS (SECONDS)
170 Nr=20          ! NUMBER OF SIGNAL-TO-NOISE RATIOS
180 DATA 0,1,2,3,4,5,6,7,8,9,10 ! INPUT SNRS IN DB
190 DATA 11,12,13,14,15,16,17,18,19
200 IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 230
210 PRINT "PROBLEM WITH PARAMETERS"
220 STOP
230 DIM Alpha(10),A(10),Beta(10),B(10),Psi(10),C(10),R(20)
240 DIM W(100),Pc(10),Ec(10),Pcv(10),Eb(10),Chb(10),Shb(10)
250 DIM Chc(10),Shc(10),Mu(10),Gamma(10),Cn(200),Ea(10)
260 DIM E(0,1000),Gca(100),Gcda(100),Gcb(100),Gbda(100)
270 DIM Cs(5000),Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
280 DOUBLE Ka,Kb,Kc,Kd,J,M,N,Nr,Ns,Ks,Ms,Kdc,Ko
290 DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
300 REDIM Alpha(1:J),A(1:J)
310 READ Alpha(*),A(*)
320 MAT Alpha=Alpha/(SUM(Alpha))
330 MAT A=(Delta)/A
340 REDIM Beta(1:M),B(1:M)
350 READ Beta(*),B(*)
360 MAT Beta=Beta/(SUM(Beta))
370 MAT B=(Delta)/B
380 REDIM Psi(1:N),C(1:N)
390 READ Psi(*),C(*)
400 MAT C=(Delta)/C
410 REDIM R(1:Nr)
420 READ R(*)
430 REDIM W(Kc:Kd)
440 CALL Weights(Kc,Kd,W(*))
450 MAT W=SQR(W)

```

```

460 REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
470 FOR Ns=1 TO N
480 C=C(Ns)
490 Pc(Ns)=Psi(Ns)*C
500 Ec(Ns)=EXP(-C)
510 NEXT Ns
520 REDIM Pcv(1:N)
530 FOR Ns=1 TO N
540 E=Ec(Ns)
550 S=0.
560 FOR Ks=1 TO N
570 S=S+Pc(Ks)/(1.-E*Ec(Ks))
580 NEXT Ks
590 Pcv(Ns)=Pc(Ns)*S
600 NEXT Ns
610 REDIM Eb(1:M)
620 FOR Ms=1 TO M
630 Eb(Ms)=EXP(-B(Ms))
640 NEXT Ms
650 REDIM Chb(1:M),Shb(1:M)
660 FOR Ms=1 TO M
670 E=Eb(Ms)
680 R=1./E
690 Chb(Ms)=.5*(R+E)
700 Shb(Ms)=.5*(R-E)
710 NEXT Ms
720 REDIM Chc(1:N),Shc(1:N)
730 FOR Ns=1 TO N
740 E=Ec(Ns)
750 R=1./E
760 Chc(Ns)=.5*(R+E)
770 Shc(Ns)=.5*(R-E)
780 NEXT Ns
790 REDIM Mu(1:M)
800 FOR Ms=1 TO M
810 Chb=Chb(Ms)
820 S=0.
830 FOR Ns=1 TO N
840 S=S+Pcv(Ns)*Shc(Ns)/(Chc(Ns)-Chb)
850 NEXT Ns
860 Mu(Ms)=Beta(Ms)*S
870 NEXT Ms
880 REDIM Gamma(1:N)
890 FOR Ns=1 TO N
900 Chc=Chc(Ns)
910 S=0.
920 FOR Ms=1 TO M
930 S=S+Beta(Ms)*Shb(Ms)/(Chb(Ms)-Chc)
940 NEXT Ms
950 Gamma(Ns)=Pcv(Ns)*S
960 NEXT Ns
970 Kdc=Kd-Kc
980 REDIM Cn(0:Kdc)
990 MAT Cn=(0.)
1000 FOR Ms=1 TO M
1010 Mu=Mu(Ms)
1020 E=Eb(Ms)
1030 P=1.
1040 Cn(0)=Cn(0)+Mu
1050 FOR Ks=1 TO Kdc
1060 P=P*E

```

```

1070   Cn(Ks)=Cn(Ks)+Mu*P
1080   NEXT Ks
1090   NEXT Ms
1100   FOR Ns=1 TO N
1110     Ga=Gamma(Ns)
1120     E=Ec(Ns)
1130     P=1.
1140     Cn(0)=Cn(0)+Ga
1150     FOR Ks=1 TO Kdc
1160       P=P*E
1170       Cn(Ks)=Cn(Ks)+Ga*P
1180     NEXT Ks
1190   NEXT Ns
1200   FOR Js=1 TO J      ! FILTER OUTPUT SIGNAL COVARIANCE
1210     A(Js)=EXP(A(Js))
1220   NEXT Js
1230   Ko=MAX(Ka,Kc)
1240   REDIM Ea(1:N),E(1:N,Ko:Kd)
1250   FOR Ns=1 TO N
1260     E=Ec(Ns)
1270     Ea(Ns)=1./E
1280     E(Ns,Ko)=EXP(-C(Ns)*Ko)
1290     FOR Ks=Ko+1 TO Kd
1300       E(Ns,Ks)=E(Ns,Ks-1)*E
1310     NEXT Ks
1320   NEXT Ns
1330   L1=MIN(Ko,Kb)+1
1340   L2=MIN(Kd,Kb)+1
1350   REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
1360   L=Kd-Ko+1
1370   REDIM Cs(1:L*(L+1)/2)
1380   L11=L1+1
1390   FOR Js=1 TO J
1400     A1=Alpha(Js)
1410     A=A(Js)
1420     A1=A*A-1.
1430     FOR Ns=1 TO N
1440       C=Ea(Ns)
1450       Ca=C*A
1460       Cda=C/A
1470       Cal=1.-Ca
1480       Hc=A-C
1490       Ae=A1*Pc(Ns)
1500       Gca=Ca^Ka/Cal
1510       Gca(L1)=Ca^L1/Cal
1520       Gcda(L1)=Cda^L1/(1.-Cda)
1530       FOR Ks=L11 TO L2
1540         Gca(Ks)=Ca*Gca(Ks-1)
1550         Gcda(Ks)=Cda*Gcda(Ks-1)
1560       NEXT Ks
1570       FOR Qs=1 TO N
1580         B=Ea(Qs)
1590         Aee=Ae*Pc(Qs)
1600         Ba=B*A
1610         Bda=B/H
1620         Cal=1.-Ba
1630         Cal1=1.-Cb
1640         Gba=Pa^Ka/Bal
1650         Aee=H-C
1660         B=B*A1*(H+Ba1)

```

```

1680 S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1690 Gcb(L1)=Cb^L1/Cb1
1700 Gbda(L1)=Bda^L1/(1.-Bda)
1710 FOR Ks=L11 TO L2
1720 Gcb(Ks)=Cb*Gcb(Ks-1)
1730 Gbda(Ks)=Bda*Gbda(Ks-1)
1740 NEXT Ks
1750 Ki=0
1760 FOR Ks1=Ko TO Kd
1770 K1=MIN(Ks1,Kb)+1
1780 S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1790 S3=Gca(K1)-Gca
1800 E=Aee*E(Ns,Ks1)
1810 FOR Ks2=Ks1 TO Kd
1820 Ki=Ki+1
1830 K2=MIN(Ks2,Kb)+1
1840 S=S2+Gbda(K2)*S3 ! SUM S
1850 Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1860 NEXT Ks2
1870 NEXT Ks1
1880 NEXT Qs
1890 NEXT Ns
1900 NEXT Js
1910 L=Kd-Kc+1 ! TOTAL WEIGHTED COVARIANCE MATRIX
1920 REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1930 L11=Kc-1
1940 R=0.
1950 FOR I=0 TO Nr
1960 Ki=0
1970 IF I=0 THEN 1990
1980 R=10.^(R(I)*.1) ! INPUT POWER SIGNAL-TO-NOISE RATIO
1990 FOR Ks1=Kc TO Kd
2000 W=W(Ks1) ! WEIGHTS (w(k))
2010 FOR Ks2=Ks1 TO Kd
2020 IF Ks1>=Ko THEN 2050
2030 Cs=0.
2040 GOTO 2070
2050 Ki=Ki+1
2060 Cs=Cs(Ki)
2070 Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
2080 L1=Ks1-L11
2090 L2=Ks2-L11
2100 Cw(L1,L2)=Cw(L2,L1)+Pr
2110 NEXT Ks2
2120 NEXT Ks1
2130 CALL Svd(L,L,Cw(*),Ca(*),D(*)) ! EIGENVALUES
2140 MAT SORT D(*) DES
2150 IF D(L)>0. THEN 2180
2160 PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
2170 PAUSE
2180 PRINT "I =";I;" CONDITION NUMBER =";D(1)/D(L)
2190 PRINT D(*)
2200 PRINT
2210 FOR Ks=1 TO L
2220 Eig(I,Ks)=D(Ks) ! STORE EIGENVALUES
2230 NEXT Ks
2240 NEXT I
2250 MASS STORAGE IS ":CS80,7"
2260 ASSIGN #1 TO "EIG"
2270 PRINT #1;Nr,Kc,Kd,Eig(*)
2280 ASSIGN #1 TO *
2290 END

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